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ABSTRACT

This handbook is designed to assist teachers of technical mathematics in developing practically-oriented curricula for their students. The underlying assumption is that, while technology students are not a breed apart, their needs and orientation are to the concrete, rather than the abstract. It describes the nature, scope, and content of curricula in Electrical Technology, Mechanical Technology, Design Drafting Technology, and Technical Physics, with particular reference to the mathematical skills which are important for the students, both in college and on the job. Sample mathematical problems, derivations, and theories to be stressed in each of these curricula are presented, as are additional materials from the physics and mathematics areas. A frame of reference is provided through discussions of the careers for which technology students are being trained. There is also a section devoted to the development of reading and study skills and to general classroom management. (Author/DC)

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Training Program for Teachers of Technical Mathematics in Two-Year Curricula

Queensborough Community College

THE CITY UNIVERSITY OF NEW YORK

BAYSIDE, NEW YORK

JC 760 420

TRAINING PROGRAM FOR TEACHERS OF TECHNICAL MATHEMATICS IN TWO-YEAR CURRICULA

A HANDBOOK PREPARED FOR THE INFORMATION OF
PARTICIPANTS IN THE SEMINAR PROGRAM HELD
AT QUEENSBOROUGH COMMUNITY COLLEGE
OF THE CITY UNIVERSITY OF NEW YORK
JUNE 7 - 18, 1976

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Preface

This handbook was written and compiled by the staff of the Training Program for Teachers of Technical Mathematics in Two-Year Curricula held at Queensborough Community College, Bayside, New York in June 1976, as part of the materials for the participants' use. The entire program was partially funded by a grant from the New York State Department of Education (Vocational Education Division) for the period July 1975-June 1976.

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Introduction

This handbook is compiled, not as a definitive guide to success in teaching technical students, but as a series of questions and suggestions, which, combined with the experiences and insights of the reader, will point the way to improved teaching, both in content and teaching methodology.

Suggestions as to course content and usable applications can at best consist of a large body of material containing more information than can be absorbed into any particular course, thereby leaving each user the option of selecting those topics that apply to specific programs. The underlying assumption is that, while technology students are not a breed apart, their needs and orientation are to the concrete, rather than the abstract. While there are differences between technology students and other students, the authors believe that in today's multi-media world, many techniques of imparting information must be developed; i.e., the classical "go home and read the book" days are gone forever.

In order to help the reader build a viable teaching plan based upon these assumptions, the authors have attempted to provide a frame of reference by discussing the careers for which technology students are being trained. The nature, scope, and content of the technical curricula will be examined, with particular reference to the mathematical skills which are important for the students, both in college and on the job. The staff of the training program have compiled a list of concrete applications of mathematics to technology, some drawn from actual textbooks currently in use in technical courses. Additional materials from the physics and mathematics areas are included. A section is devoted to the development of reading and study skills, as well as general classroom management techniques.

Development of the Technology Curriculum

In the period from 1950 to 1960, engineering colleges shifted from a practical (laboratory-oriented) curriculum to a science (theory-oriented) one. Thus the engineers graduated from these programs gravitated to industrial positions demanding a research and development background. This left a void in the engineering and scientific spectrum in the more practical area. The scientist and the engineer were at one end of this spectrum, and the craftsman was at the other end. There existed, then, a need for practical technical personnel in the area between these two ends.

It is not surprising that the demands of industry engendered the advent of a new type of program for the academic training of these newly required technical personnel. The proliferation of associate degree programs in engineering technology starting in 1960 was the inevitable result.

The kinds of institutions offering educational programs in engineering technology vary. They belong primarily in one of the following classifications:

(a) Monotechnical institutes—Single purpose institutions offering engineering technology.

(b) Polytechnical Institutions—Institutes with a variety of programs related to business, health, or public service as well as to engineering.

(c) Comprehensive Community Colleges — Community and/or junior colleges which include in their offerings various occupational-technical programs as well as "university parallel" or "transfer" programs.

(d) Universities—Senior institutions which include associate degree programs in engineering technology as part of their offerings.

Engineering technology is concerned primarily with the application of established scientific and engineering knowledge and methods. Normally, engineering technology is not concerned with the development of new principles and methods. Technical skills such as drafting, in one instance, are characteristic of engineering technology.

The associate degree graduate of an engineering technology program is called an engineering technician. The graduate of a baccalaureate program in engineering technology is known as an engineering technologist. The engineering technologist has academic training which lies between that for the engineering technician and the engineer.

The primary objective of the associate degree program, therefore, is to turn out a marketable engineering technician who will take his or her place in the industrial world.

The Engineers' Council for Professional Development (ECPD) is the professional society which accredits individual engineering technology programs and therefore sets minimum criteria standards.

To meet minimum criteria for ECPD accreditation, an associate degree program in engineering technology should include the following four major curriculum subdivisions:

(a) **Technical Studies**, which include the major technical specialties, related technical studies and technical sciences, for about one year.*

(b) **Basic Science Studies**, which include mathematics, applied mathematics and the physical sciences, for about half a year.*

(c) **Non-Technical Studies**, which include communications, humanities, social sciences, and other life-oriented subject matter, for about a third of a year.*

(d) **Institutional Electives**, which may include additional technical, basic science, or non-technical studies or other content considered necessary to maintain the integrity or achieve the special purpose of an institution of higher education, sufficient to make the total two years.*

As a prime consideration, theory courses in the technical specialties should be accompanied by coordinated laboratory experience which stresses measuring physical phenomena and collection, analysis, interpretation and presentation of data. Students should be reasonably familiar with modern types of apparatus that they may encounter in industry.

In the industrial world, the engineering technician's function is not single job-oriented; he or she may have to move into many different roles.

In general, the engineering technician works with the scientist, engineer and technologist, assisting them in the practical aspects of their efforts, and directing the arts and skills of the craftsman. He differs from the craftsman in his knowledge of engineering theory and methods, and he differs from the engineer in his more specialized technical background and specialized technical skills. The engineering technician utilizes a combined variety of skills and diversified practical and theoretical knowledge to get things done.

Some of the major technical and engineering areas in which an engineering technician may work include the following:

(a) Maintenance—the continuation of a system or sub-system so that it may meet the specifications as originally envisioned and for which design criteria were established.

(b) Production or Manufacturing—the technician may provide technical supervision in installation, start-up, checkout of equipment and systems; trouble-shoot and diagnose malfunctions in laboratory prototypes or production equipment, systems of processes. In addition, he or she may monitor product quality and develop schedules for work flow including all operations from raw materials to finished products.

(c) Testing—the testing of equipment, materials, and processes to determine whether they meet specifications and accepted engineering standards.

(d) Technical Sales—attempts are made to convert the needs (often vague) of potential customers into systems that will satisfy them and at the same time produce business for the company that the technician represents.

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* *One year of academic time is considered to be a minimum of 30 semester credits. From the above criteria, the minimum number of semester credits for an associate degree in engineering technology would be 60. However, most programs range in semester credits between 64 and 72.*

(e) Technical Writer—the technician works with engineers in compiling technical manuals, reports, bulletins, specifications and catalogs.

(f) Estimating—work with engineering firms, building supply companies, and others in preparing cost estimates from drawings and specifications.

(g) Design—concerned with implementing the specifications of the customer, then combining materials, involving a variety of processes, into a finished product or system.

(h) Development—provides the bridge between the design function and that of applied science. The development engineer or technician puts together a prototype system or sub-system that may meet only a part of the specification for an engineering endeavor.

(i) Research—the technician works with scientists and engineers to develop new equipment and to evolve new applications in manufacturing processes.

(j) Field—technicians are concerned with placing the system in operation under actual operating conditions.

(k) Supervision—supervision of lower-grade technicians and skilled craftsmen.

The paramount distinction between an engineering technology program and a vocational or industrial technology program is the level and quality of the basic sciences—specifically the mathematics. It is commonly stated that mathematics is one of the more critical determinants of both the level and quality of an engineering technology program. The McGraw Report in 1962 conducted by the American Society for Engineering Education (ASEE) and titled "Characteristics of Excellence in Engineering Technology Education" contained the recommendation that an associate degree in engineering technology contain a minimum of 9 semester credits in mathematics (algebra, trigonometry, calculus). The language of the recommendation encouraged institutions to exceed the stated minimum.

Contemporary thinking is largely unchanged. There is, however, some opposition to the compulsory inclusion of calculus (an ECPD minimum criterion required for accreditation) in that some engineering technology programs—specifically in the industrial major— could better utilize a course in statistics instead of an exposure to calculus. The 1972 ASEE report titled "Engineering Technology Education Study" retained the spirit of the McGraw Report but attempted to make quantitative mathematics requirements more adaptable to the needs of individual institutions.

Educational philosophy, in the case of a curriculum's mathematical content, has been translated directly into evaluative criteria for accreditation. The ECPD, responsible for the accreditation of engineering technology programs, has published the following statement:

> "An engineering technology curriculum acceptable to ECPD will normally be characterized by at least the equivalent of one-half academic year of basic sciences, about half of which is mathematics and of which the mathematics includes carefully selected topics suited to each curriculum from appropriate areas of mathematics beyond algebra and trigonometry, and including basic concepts of the calculus."

Institutions vary considerably in the matter in which they implement mathematics instruction in their programs. Some, for example, neither teach mathematics as a discipline nor present it separately in formal courses; rather, they attempt to introduce mathematical concepts as part of the technical specialty at the time such concepts become needed, integrating them completely with the technical specialty. It is difficult to refute arguments that mathematics presented in such a way becomes more meaningful to students because it has an immediate application and relevance. This method is not used by the majority of institutions.

In institutions where separate mathematics courses are given, the faculty which teach these courses fall into three separate categories.

(a) A separate "liberal arts" mathematics department whose faculty service the technology departments by teaching the math courses.
(b) Engineering technology faculty (engineers) who teach the math courses to the technical students.
(c) Math teachers (with degrees in mathematics) who are members of the engineering technology division.

The biggest stumbling block, as concerns the mathematics academic welfare of technology students, is the liberal arts oriented math faculty in group (a). They lack the introspection necessary to present "technical" mathematics properly. They either are completely lacking in interest, do not have the proper background, or insist that the traditional "liberal arts" approach is best. At best their motives are suspect because the proper teaching of these mathematics courses involves additional work.

To insure proper handling of these courses, both in level and content, a continuous dialogue must be present between the mathematics and technology departments.

The most frequently encountered course pattern is a sequence of three courses—algebra, trigonometry, and elements of calculus; considerable variation in nomenclature exists, however, and is to be expected. In most cases, remedial courses are offered, without credit, for the benefit of those students who lack prerequisites.

The content of mathematics courses in engineering technology should be slanted towards operations and applications rather than toward theory and derivations.

In an extensive survey of mathematics faculty members teaching at institutions with technology curricula, it was found that they gave emphasis to the following topics:

(a) **Algebra** — The topical areas receiving substantial emphasis included fundamental operations, special products, factoring, fractions, exponents and radicals, linear and fractional equations, systems of equations, elementary determinants and logarithms. Moderate emphasis was given to the nature of number systems, functions and graphs, complex numbers, equations of the third or higher degree, inequalities, ratio and proportions and progressions. Little or no emphasis was usually given to such aspects of algebra as mathematical induction, binomial theorem, permutations and combinations, probability, determinants of higher order and partial fractions.

(b) **Trigonometry** – Almost all the usual topics in plane trigonometry – those appearing in standard introductory textbooks – were given substantial emphasis, although work with certain topics such as identities, circular functions, inverse functions and application of the law of tangents, were said to be often curtailed because of the time restriction. Solutions of right and oblique triangles were said to be given special emphasis.

(c) **Calculus**–Emphasized most strongly are coordinate systems, lines, variables and functions, limits, differentiation, applications of the derivative, integration of algebraic forms, and integration of simple transcendental forms. Receiving moderate emphasis are topics such as applications of integration, differentiation of special forms, differentiation with respect to time curve tracing and equations of the second degree. Areas that are said to receive little or no emphasis include parametric equations, polar coordinates, indeterminate forms, infinite series, expansion of functions, hyperbolic functions, solid analytic geometry, multiple integrals, vector analysis and differential equations. However, it should be pointed out that many of these last topics are included when four courses rather than three constitute the mathematics sequence. This more "advanced" calculus content is usually in the electrical technology programs. The graduate of this program is normally more "sophisticated" in the use of the mathematics because of the nature of the desired electrical content of the more advanced technical courses.

But the need to remedy the entering students' deficiencies should not blind us to an equally important problem. Every entering student with deficiencies must be motivated to continue his or her studies in engineering technology. This motivation, a reinforcement of the original choice of career, cannot begin too soon. Taking remedial courses for a semester or longer (without any technical specialty courses) dampens the student's enthusiasm for his newly-chosen profession. The best incentive (during this period) is for the student to take concurrently a practically oriented hands-on course in the technology area. This helps to sustain the engineering technology student's interest in his remedial work.

The Queensborough Community College Program in Electrical Technology

As an example of a two-year program in Electrical Technology, the courses and course descriptions for the technical specialty and mathematics at Queensborough Community College are described below.

2. ELECTRICAL TECHNOLOGY (Electronic)—A.A.S. REQUIRED PROGRAM

(AN ECPD [ENGINEERS COUNCIL FOR PROFESSIONAL DEVELOPMENT] ACCREDITED ENGINEERING TECHNOLOGY CURRICULUM)

UAPC Code for Electrical Technology (Electronic Option)—Day: 0923; Evening: 3923

Semester I		*Credits*
*MA-14	*Technical Mathematics A	4
SS- *or* HI-	Elective in Social Science *or* History	3
*EN-11	*English Composition I	3
ET-11	Electric Circuit Analysis I	4
HE-101 *or* 102	Health Education	1-2
PE-300, 400, 500, *or* 700 series (excluding PE-711, 712, 713, 714, 760, 761)	Physical Education or Dance	1
	Sub-total	**16-17**
Semester II		
MA-15	Technical Mathematics B	3
PH-201 (formerly 20)	General Physics I	4
ET-12	Electric Circuit Analysis II	4
ET-21	Electronics I	4
	Sub-total	**15**
Semester III		
PH-202 (formerly 21)	General Physics II	4
MA-16	Technical Mathematics C	2
ET-13	Transient Circuit Analysis	3
ET-22	Electronics II	4
ET-31	Electrical Machinery	4
ET-41	Electronic Project Laboratory	1
	Sub-total	**18**
Semester IV		
SS- *or* HI-	Elective in Social Science *or* History	3
ET-23	Communications and Microwave Electronics	4
ET-32	Feedback Control Systems	4
ET-51	Digital Computers	4
EN-12	English Composition II	3
	Sub-total	**18**
	TOTAL ELECTRICAL TECHNOLOGY	**67-68**

** Satisfactory score on placement examinations required.*

Courses Taught in Electrical Technology

ET-01. INTRODUCTION TO ELECTRICAL TECHNOLOGY
1 class hour 2 recitation hours 3 laboratory hours 0 credits
Prerequisite or corequisite: MA-10. Required for all Electrical Technology majors taking MA-10.

Introduction to electronic and computer technology, scientific notation, electrical units, schematic electrical diagrams, fundamentals of computers, Ohm's law, electrical components, and measuring instruments. Laboratory hours complement class work.

ET-11. ELECTRIC CIRCUIT ANALYSIS I
3 class hours 3 laboratory hours 4 credits
Prerequisite: ET-01 for those students required to take MA-10 as a result of mathematics placement examination. Corequisite: MA-14.

Resistance; Ohm's law; Kirchhoff's laws, networks with DC current and voltage sources; branch-current analysis; mesh and nodal analysis, superposition, Thevenin's, Norton's, maximum power theorems; capacitance; magnetic circuits; inductance; DC meters. Laboratory hours complement class work.

ET-12. ELECTRIC CIRCUIT ANALYSIS II
3 class hours 3 laboratory hours 4 credits
Prerequisite: ET-11. Corequisite: MA-15.

Sinusoidal waveforms; phasor quantities; impedance; Kirchhoff's laws; network theorems; power; resonance; three phase circuits; AC meters; harmonics; mutual inductance. Laboratory hours complement class work.

ET-13. TRANSIENT CIRCUIT ANALYSIS
1 class hour 2 recitation hours 2 laboratory hours 3 credits
Prerequisite: ET-12. Corequisite: MA-16.

The differential equation formulation of electric circuit behavior; the forced solution; the source-free solution; initial conditions; the complete solution; complex frequency; transfer function; pole-zero concept; Laplace transform. Laboratory hours complement class work.

ET-21. ELECTRONICS I
3 class hours 3 laboratory hours 4 credits Corequisite: ET-12 or ET-14.

Basic theory and operation of solid-state and vacuum tube devices including diodes, triodes, pentodes, transistors, field-effect transistors, unijunction transistors, silicon-controlled rectifiers, tunnel diodes, varactors and Zener diodes. Clipping and clamping circuits. Graphical and equivalent circuit analysis of active devices. Biasing of transistors. Rectifier, filter and power supply circuit design. The major emphasis throughout the course is on semiconductor devices. Laboratory hours complement class work.

ET-22. ELECTRONICS II
3 class hours 3 laboratory hours 4 credits Prerequisite: ET-21.

Hybrid parameters, design of small and large signal amplifiers (transistor, FET and vacuum tube); decibels; frequency response of amplifiers; D.C. amplifiers, operational amplifier circuits; integrated circuit theory; regulated transistor power supplies; SCR and triac control circuits; unijunction transistor circuits. Laboratory hours complement class work and include a design project.

ET-23. COMMUNICATIONS AND MICROWAVE ELECTRONICS
3 class hours 3 laboratory hours 4 credits Prerequisite: ET-22.

Generation and processing of signals, including oscillation, modulation, demodulation, frequency conversion; bandwidth and noise; transmission lines and waveguides; use of the Smith chart; tuned circuits; transmission line sections; microwave cavities; microwave generators and amplifiers, including klystrons, magnetrons and traveling wave tubes. Lasers and masers. Laboratory hours complement class work.

ET-31. ELECTRICAL MACHINERY
3 class hours 3 laboratory hours 4 credits Prerequisite: ET-12 or 14.

Characteristics and applications of DC motors and generators; transformers; AC motors and generators; motor starters and control; power factor correction; power systems. Laboratory hours complement class work.

ET-32. FEEDBACK CONTROL SYSTEMS

3 class hours 3 laboratory hours 4 credits Prerequisite: ET-13, 31.

Analog computation and simulation; the block diagram concept voltage and speed control systems; servo components and transducers; second order servomechanisms; proportional control and tach feedback; frequency response analysis using the Bode Plot; stability; instrument servos, digital servos and computer control; pneumatic and hydraulic systems. Laboratory hours complement class work.

ET-41. ELECTRONIC PROJECT LABORATORY

3 laboratory hours 1 credit Corequisite: ET-22.

A practical course in the use of the tools of the electronic technician; techniques are developed by building and testing electronic equipment such as a transistorized superheterodyne radio; layout and development of printed circuits.

ET-51. DIGITAL COMPUTERS

3 class hours 3 laboratory hours 4 credits Prerequisite: ET-22.

Number systems; Boolean algebra; memory elements; logic elements; timing elements; digital computer logic circuits—AND, OR, NAND, NOR; multivibrator circuits—flip-flop, clock, one-shot; computer organization-arithmetic, control, memory, input and output units; elements of programming. Laboratory hours complement class work.

Mathematics

MA-10. BASIC MATHEMATICAL SKILLS *5 class hours 0 credits*

Review of arithmetic, geometric notions, beginning algebra. *(Note: a modular approach is used in day classes).*

MA-14. TECHNICAL MATHEMATICS A *4 class hours 4 credits*

Prerequisite: MA-10, or satisfactory score on mathematics placement examination.

A basic presentation of the fundamental concepts of college algebra and trigonometry with scientific and engineering applications; linear equations and systems; coordinate geometry and functions; quadratic equations.

MA-15. TECHNICAL MATHEMATICS B

3 class hours 3 credits Prerequisite: MA-14, or the equivalent.

A continuation of Technical Mathematics A (MA-14): trigonometric, exponential, and logarithmic functions and their graphs; complex numbers with applications to vector problems; elements of analytic geometry, curve sketching; introduction to differential and integral calculus.

MA-16. TECHNICAL MATHEMATICS C

2 class hours 2 credits Prerequisite: MA-15, or the equivalent.

A continuation of Technical Mathematics B (MA-15): basic elements of differential and integral calculus and their applications; conic sections.

Job titles for graduates of an associate degree program in Electrical Technology include: electronic technician, computer technician, electrical designer, technical sales representatives, technical writer, electrical estimator, electrical contractor, research technician, communication technician, customer-service technician, electrical engineering aide, medical electronics technician, broadcast technician, field engineering assistant, process central technician and test equipment technician.

Introduction to Electrical Technology

The Department of Electrical Technology offers the remedial course, "Introduction to Electrical Technology" (3 class hours, 3 laboratory hours, no credit) which must be taken concurrently with the remedial math course, MA-10, "Basic Mathematical Skills." This course (which is 5 class hours, no credit) includes a review of arithmetic, geometric notions and beginning algebra. The remedial ET course, "Introduction to Electrical Technology" has a two-fold objective: (1) To complement and reinforce MA-10 through practical applications in the field of electrical technology, and (2) to motivate the student in his original choice of a career in electrical technology.

The ET-01 recitation portion of the course includes a mathematical review (12 hours) of the following areas: positive and negative numbers, fractions, decimals, scientific notation, squares and square roots, laws of exponents and solution of simple algebraic equations. The use of the slide rule for multiplication and division is also included. The remaining 30 hours of recitation covers elementary electrical theory and shows the application of the mathematics in the technical area. The laboratory work (42 hours) complements the recitation theory and includes films on the slide rule as well as on the various areas of electrical theory

Some typical mathematical problems and derivations and theory include:

Fractions:

1. $\frac{5}{8} - \frac{2}{3} + \frac{1}{4} - \frac{1}{2} =$

2. $(\frac{3}{4})(-\frac{1}{2})(-\frac{1}{3}) =$

3. $(\frac{2}{3} - \frac{1}{9}) \div (\frac{3}{4} + 2) =$

4. $\dfrac{\frac{2}{3} + \frac{4}{5}}{\frac{1}{2} + \frac{3}{4}} =$

Changing Fractions to Decimals (round off to 3 decimal places):

5. $\frac{1}{8} =$

6. $\frac{9}{100}$

7. $\frac{11}{64} =$

Scientific Notation

Write the following numbers in scientific notation:

8. 0.0000316

9. 3,240,000

Powers of 10

Express the following as powers of 10:

10. 10,000,000

11. 0.00001

12. $10^{10} \times 10^{-7} \times 10^{3}$

Express each of the factors as a power of 10 and find the product:

13. 1,000,000 x 10,000

14. 100,000 x 0.0001

Carry out the following divisions:

15. $\frac{10^5 \times 10^{-3}}{10^{-2}}$

16. $\frac{10^0 \times 10^{-4}}{10^3}$

Change to decimal form:

17. 826×10^{-6}

18. 0.003×10^{-4}

Calculate the following:

19. $\frac{0.0000004 \times 5{,}000{,}000 \times 2{,}000}{5{,}000 \times 0.0002 \times 20{,}000}$

20. $\frac{600{,}000 \times 0.0004 \times 0.02 \times 0.0000007}{0.0005 \times 1{,}000 \times 20 \times 200{,}000{,}000}$

Find the answers to the following using the slide rule:

21a. $(0.13)^2$

21b. $(0.013)^2$

21c. $(0.00013)^2$

Find the answers to the following using the slide rule:

22a. $\sqrt{1700}$

22b. $\sqrt{0.00017}$

22c. $\sqrt{17{,}000}$

The use of the prefixes milli (10^{-3}), micro (10^{-6}), kilo (10^3) and mega (10^6) is of great importance in technology in conversions.

Conversions using the above are a source of great confusion to the student. They appear to have no difficulty in converting dollars to cents or feet to inches since in "real" life they have been exposed to these quantities. It is stressed to the student that conversion of a large unit to a smaller unit will result in a bigger number (example: $3.75 = 375 cents and visa versa).

1 ampere = 1,000 milliamperes = 10^3 ma
1 milliampere = 0.001 amperes = 10^{-3}a
1 volt = 10^6 microvolts = 1,000,000 microvolts
1 microvolt = 10^{-6} volts = 0.000001 volts
1 kilovolt = 1,000 volts = 10^3 volts
1 volt = 0.001 Kilovolts = 10^{-3} kilovolts
1 megavolt = 1,000,000 volts = 10^6 volts
1 volt = 0.000001 megavolts = 10^{-6} megavolts

Intuitively then, if 250 milliamperes were to be converted to amperes, the result should be a number smaller than 250. Conversely, if 3 kilovolts were to be converted to volts, the resultant number should be greater than 3.

Conversions may be accomplished mathematically as follows:

Convert 250 milliamperes (ma) to amperes (a):

$$250 \text{ ma} \times \frac{1\text{a}}{1{,}000\text{ma}} = \frac{250 \times 1\text{a}}{1{,}000} = 0.25\text{a}$$

which simply means division by 1,000.

Convert 3.2 kilovolts to volts.

$$3.2 \cancel{\text{KV}} \times \frac{1{,}000 \text{ V}}{1\cancel{\text{KV}}} = 3{,}200 \text{ volts}$$

Convert 34 milliohms ($m\Omega$) to micro ohms ($\mu\Omega$)

$$34 m\Omega \times \frac{1{,}000 \; m\Omega}{1 \; m\Omega} = 34{,}000 \text{ micro-ohms}$$

In the above example since the micro-ohm is a smaller unit than the milliohm, the resultant answer in micro-ohms should be larger than 34.

Make the following conversions

22 milliamperes = amperes
0.002 amperes = milliamperes
0.423 amperes = microamperes
426 milliamperes = microamperes
0.05 kilovolts = volts
78 volts = millivolts
0.5 volts = microvolts
500 millivolts = volts

A famous law of nature indicates that effect is equal to the cause divided by the opposition. Mathematically:

$$\text{effect} = \frac{\text{cause}}{\text{opposition}} \quad \text{(eq. 1)}$$

In an electrical circuit the effect is the current, I, the cause is the voltage, V, and the opposition is the resistance, R. Therefore for the electric circuit, eq. 1 (known as Ohm's Law) can be expressed by:

$$\text{(eq. 2)} \quad I = \frac{V}{R} \quad \text{or} \quad R = \frac{V}{I} \quad \text{or} \quad V = IR$$

In technology, units are of great importance - there appears to be a dearth of this principle in math courses where all operations appear to be dimensionless (certainly not the case in "real" life).

In the various forms of equation 2, I must be in amperes, V in volts and R in ohms (Ω).

Examples:

A carbon filament lamp draws a current of 0.5 amperes when a voltage of 120 volts is applied. What is its resistance?

$$R = \frac{V}{I} = \frac{120\text{v}}{0.5\text{a}} = 240 \text{ ohms } (\Omega)$$

A 6 volt battery is connected to a resistance of 2 kilo-ohms. How much current is will be delivered?

2 kilo-ohms is equal to 2,000 ohms.

$$I = \frac{V}{R} = \frac{6 \text{ Volts}}{2{,}000} = 3(10)^{-3} \text{ amperes or 3 milli-amperes}$$

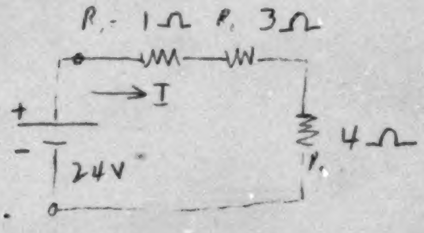


Fig 1 - Series Circuit

In a series circuit the total resistance, R_T, is the sum of the individual resistors in series. Find the current that flows in figure 1.

$$R_T = R_1 + R_2 + R_3 = 1 + 3 + 4 = 8\,\Omega$$

$$I = \frac{V}{R_T} = \frac{24 \text{ Volts}}{8\,\Omega} = 3 \text{ amperes}$$

In the above circuit find the electrical power, in watts, dissipated in the 4 ohm resistor.

$$\text{Power } (P) = I^2R$$

$$\therefore\ P = (3)^2 4 = 9 \times 4 = 36 \text{ watts}$$

In an electrical circuit, the power consumed in a resistor of 2 ohms is 72 watts. Find the current in the resistor.

$$P = I^2R \qquad I^2 = \frac{P}{R}$$

$$I = \sqrt{\frac{P}{R}} = \sqrt{\frac{72}{2}} = \sqrt{36} = 6 \text{ amperes}$$

In the following electrical equations, solve for the quantity indicated.

$I = \frac{Q}{t}$ solve for t, solve for Q $\qquad R = \frac{pL}{A}$ solve for L, solve for A

$P = I^2R$ solve for R, solve for I $\qquad W = \frac{LI^2}{2}$ solve for I, solve for L

$L = \frac{N^2 A}{1}$ solve for N, solve for 1

$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$ solve for R_T, solve for R_1

$K = \frac{M}{\sqrt{L_1L_2}}$ solve for M, solve for L_1

$Z^2 = R^2 + X^2$ solve for X $\qquad \frac{T + t_1}{R_1} = \frac{T + t_2}{R_2}$ solve for R_1

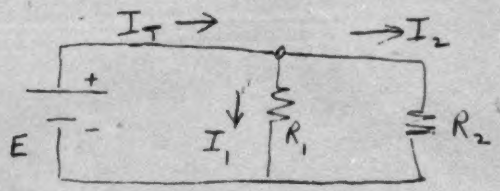


Figure 2 – Parallel Connection of 2 Resistors

Derive the equation for the total resistance R_T, in terms of R_1 and R_2.

1) $R_T = \frac{E}{I_T}$

2) $I_T = I_1 + I_2$

3) $I_1 = \frac{E}{R_1}$

4) $I_2 = \frac{E}{R_2}$

5) $I_T = \frac{E}{R_1} + \frac{E}{R_2}$

6) $R_T = \frac{E}{\frac{E}{R_1} + \frac{E}{R_2}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$

7) $R_T = \frac{1}{\frac{R_2 + R_1}{(R_1)(R_2)}} = \frac{R_1 R_2}{R_1 + R_2}$

which indicates that the total resistance of two resistors in parallel is the product over the sum.

Further, prove that the value of R_T will always come out less than the value of the smaller of the two resistors.

$$R_T = \frac{R_1 R_2}{R_1 + R_2}$$

Let R_2 be the smaller of the two resistors. Dividing the numerator and denominator of the right hand side of the equation for R_T by R_1 yields:

$$R_T = \frac{\dfrac{R_1 R_2}{R_1}}{\dfrac{R_1}{R_1} + \dfrac{R_2}{R_1}} = \frac{R_2}{1 + \dfrac{R_2}{R_1}}$$

Since R_2 and R_1 are positive numbers, the denominator of the right side of the equation is greater than 1. Therefore R_2 divided by a number greater than 1 gives a value for R_T which is less than R_2.

Example:

A 3 ohm and 6 ohm resistor are in parallel. What is the value of the total resistance?

$$R_T = \frac{3 \times 6}{3 + 6} = \frac{18}{9} = 2 \text{ ohms}$$

Note that this value is smaller than the smaller resistor of 3 ohms. This will serve as a check on the student's work. If the answer were greater than 3 ohms, then the student would know that his or her calculation was incorrect.

If more than 2 resistors are in parallel, then the total resistance, R_T, is given by:

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \text{———}$$

Example:

Three resistors are in parallel. Their values are respectively 2 ohms, 5 ohms, and 10 ohms. What is the total resistance of this combination?

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \qquad \frac{1}{R_T} = 0.5 + 0.2 + 0.1 = 0.8$$

$$\frac{1}{R_T} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} \qquad R_T = \frac{1}{0.8} = 1.25 \text{ ohms}$$

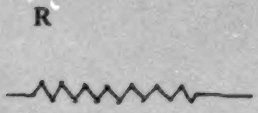
Note again that the answer is less than the value of the smallest resistance of 2 ohms.

The mathematical and electrical theory of resistors in series and in parallel, as well as Ohm's Law, are checked experimentally in the laboratory.

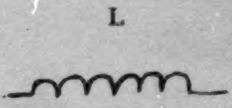
Electrical Circuit Analysis

The first college credit electrical course is Electric Circuit Analysis I (ET-11), which consists of 3 class hours, 3 laboratory hours, and carries four credits. The course involves the solution of electrical circuits with a constant (direct current) input. For a constant (dc) input, the mathematics in the course is simplified. It must be remembered that the non-remedial math student takes this course in the first semester. Corequisite with this course is Technical Math A (MA-14). The MA-14 course includes fundamentals of college algebra and trigonometry, linear equations and systems, coordinate geometry and functions and quadratic equations.

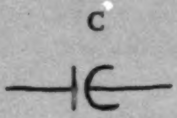
The ET-11 course deals with the circuit applications of the basic three electrical elements. They are the resistance, R; the inductance, L; and the capacitance, C. The symbols are indicated below.



unit - ohms (Ω)



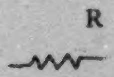
unit - henry (H)



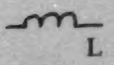
unit - farad (F)

The resistance is an electrical energy dissipating element while the other two elements store electrical energy. The resistance "behaves" quite well mathematically (there is a linear relationship between the voltage [v] across it and the current [i] through it). The energy storing elements, L and C, behave "abominably" in the mathematical sense since the relationship between voltage and current in these elements is governed by differential and integral equations.

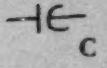
The volt-ampere (cause and effect) relationships are given below:



$$v = iR \qquad i = \frac{v}{R}$$



$$v = L\frac{di}{dt} \qquad i = \frac{1}{L}\int v\,dt$$



$$v = \frac{1}{C}\int i\,dt \qquad i = C\frac{dv}{dt}$$

Note that the symbol "t" represents time.

It is stressed to the students again, that, although in their math course the ubiquitous variables will be "x" and "y", in the electrical world, these variables will be noted by their absence. Their replacements will include v, i, t, q, w, p and many others.

More than three-quarters of the course is devoted to the behavior of electrical circuits (using various theorems) to the resistance, R, because of the simpler mathematics involved. The latter portion of the course, dealing with L and C elements, will result in the introduction of the students to simple concepts of the derivative and the integral. That the ET faculty must integrate concepts of mathematics not yet covered in the concurrent mathematics course is a fact of life that cannot be overcome.

As indicated before, emphasis is again placed on conversion of units, solution of algebraic equations, scientific notation, slide rule (although this instrument may soon be rendered obsolete by the calculator) and dimensional analysis.

Basic equations in the course include:

$$R = \frac{V}{I} \qquad I = \frac{Q}{t} \qquad P = VI \qquad P = I^2R$$

$$W = VIt \qquad \text{eff} = \frac{P_o}{P_{in}} \times 100 \qquad R_T = \frac{R_1 R_2}{R_1 + R_2}$$

$$R = \rho \frac{L}{A} \qquad \frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \text{------}$$

$$\frac{R_1}{R_2} = \frac{T + t_1}{T + t_2} \qquad V_1 = \frac{E R_1}{R_T}$$

In the laboratory, the basic concepts of plotting a graph from data obtained is stressed. The introduction of the independent variable as the abscissa and the dependent variable as the ordinate is introduced. A graph sheet, properly identified with units on both axes and with a proper title, should be able to "stand" by itself – that is, if a person looked only at that sheet, he need not refer to any other information.

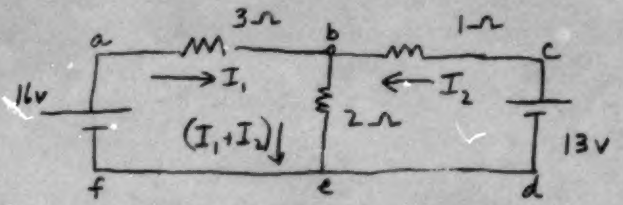
The solution of two (sometimes three) linear simultaneous equations, using determinants, is essential. It is pointed out to the student that, in industry, two equations, as indicated below, will never be encountered.

$$2x - y = 6 \qquad\qquad -4x + 3y = 8$$

In the math course, the student is taught to solve the equations by multiplying the top one by 2 and then by adding, a solution for "y" is immediately forthcoming.

In real life, the coefficients are never that simple and, in addition, any self-respecting engineering technician will always use determinants.

Example:



In the above circuit, solve for the current in each resistor.

Assuming current directions as indicated above, Kirchhoff's law of voltage is applied to each of two loops

loop a b e f a – $16 = 3I_1 + 2(I_1 + I_2)$

loop c b e d c – $13 = 1I_2 + 2(I_1 + I_2)$ $\quad 16 = 5I_1 + 2I_2 \quad 13 = 2I_1 + 3I_2$

$$I_1 = \frac{\begin{vmatrix} 16 & 2 \\ 13 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 2 \\ 2 & 3 \end{vmatrix}} = \frac{16(3) - 2(13)}{5(3) - 2(2)} = \frac{22}{11} = 2 \text{ amp.}$$

$$I_2 = \frac{\begin{vmatrix} 5 & 16 \\ 2 & 13 \end{vmatrix}}{11} = \frac{5(13) - 2(16)}{11} = \frac{33}{11} = 3 \text{ amp.}$$

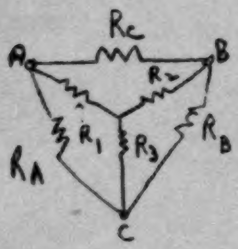
$I_1 + I_2 = 2 + 3 = 5$ amp.

Therefore, the current in the 3Ω resistance is 2 amperes, in the 1Ω resistance it is 3 amperes, and the current in the 2Ω resistance is 5 amperes.

The mathematical solution to this problem would be verified experimentally in the laboratory.

If an electric circuit has three loops then a third order determinant would result from the electrical principles.

Example:



Given the 3 resistances, R_A, R_B, and R_C in a delta connection and R_1, R_2, and R_3 in a wye connection

(a) Solve for R_1, R_2, R_3 in terms of R_A, R_B, R_C

(b) Solve for R_A, R_B, R_C in terms of R_1, R_2, R_3

Using electrical theory at points A – C

$$R_1 + R_3 = \frac{R_A(R_B + R_C)}{R_A + R_B + R_C}$$

points b-c give:

$$R_2 + R_3 = \frac{R_B(R_A + R_C)}{R_B + R_A + R_C}$$

points a-b give:

$$R_1 + R_2 = \frac{R_C(R_A + R_B)}{R_C + R_A + R_B}$$

Solving the above 3 simultaneous equations yields for R_1, R_2, R_3:

$$R_1 = \frac{R_A R_C}{R_A + R_B + R_C} \qquad R_2 = \frac{R_B R_C}{R_A + R_B + R_C} \qquad R_3 = \frac{R_A R_B}{R_A + R_B + R_C}$$

Solving for R_A, R_B, R_C yields:

$$R_A = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_2} \qquad R_B = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_1} \qquad R_C = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_3}$$

In the case of a balanced system (identical loads in the wye or delta connection), then it would follow that:

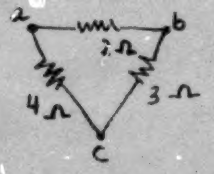
$$R_1 = R_2 = R_3 = R_y = \frac{R_\Delta^2}{3R_\Delta} = \frac{R_\Delta}{3}$$

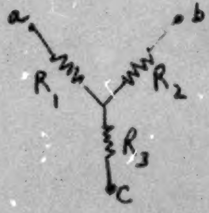
In a similar manner then:

$$R_A = R_B = R_C = R_\Delta = \frac{3R_Y^2}{R_Y} = 3R_Y$$

Problems:

1) Convert the delta system below to an equivalent wye system:



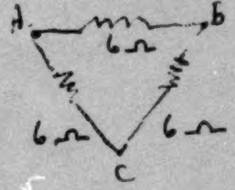


$$R_1 = \frac{2 \times 4}{2 + 3 + 4} = \frac{8}{9}\,\Omega$$

$$R_2 = \frac{2 \times 3}{2 + 3 + 4} = \frac{6}{9} =$$

$$R_3 = \frac{3 \times 4}{2 + 3 + 4} = \frac{12}{9}$$

2) Convert a balanced wye system, each of whose loads are resistors of 2 ohms, to the eequivalent balanced delta



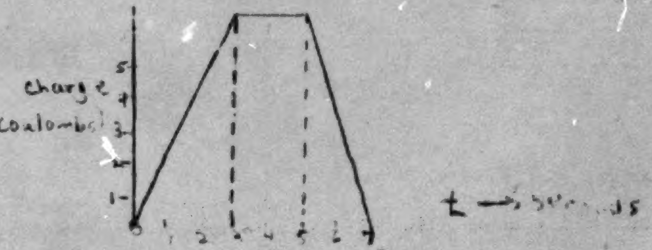
$$R_A = R_B = R_C = R_\Delta = 3(2) = 6$$

At the end of this first semester electrical course, the elementary concepts of the derivative and the integral are introduced even though the electrical student will not cover this material until the second semester math course (Technical Math B – MA-15).

The derivative concept is introduced as the slope and the integral as the area under the curve (with respect to the horizontal axis) between two limits.

Problem:

If the charge, q, in coulombs, and the current, i, in amperes, are related through the equation, i = dq/dt, where t is the time in seconds, sketch the curve of the current, i, if the charge curve is given below:



From 0 to 3 seconds the slope of the tangent to the q curve is constant at

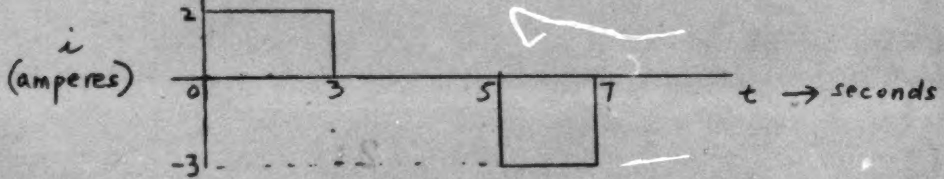
$$\frac{6}{3} = 2\ \frac{\text{coulombs}}{\text{second}}$$

From 3 to 5 seconds, the slope is 0 and from 5 to 7 seconds the slope is

$$-\frac{6}{2} = -3\ \frac{\text{coulombs}}{\text{second}}\ \text{(amperes)}$$

From 7 seconds on the slope is again zero.

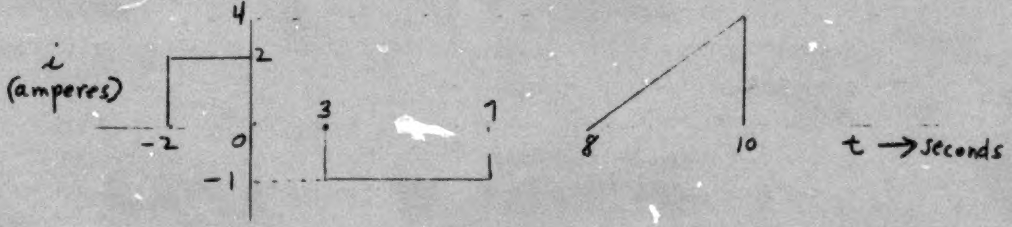
The dq/dt or i curve is shown below



It is pointed out to the students that the curves of two variables will have the identical shape only if the variables are linearly related.

Problem

The current, i, curve is shown below. Find the charge q, at t = 10 seconds if $q = \int_{-\infty}^{t} i\,dt$

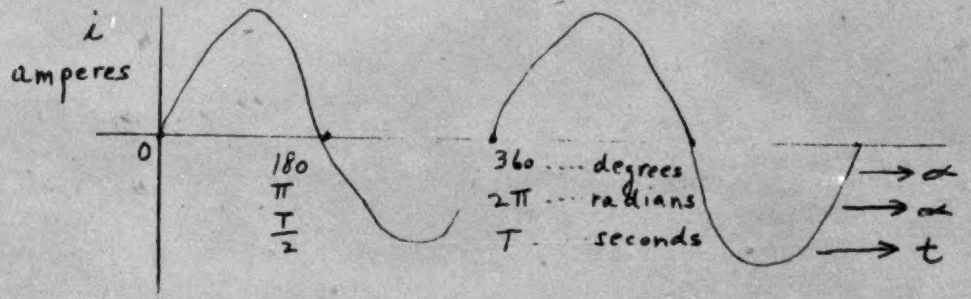


The value of the charge, q, in coulombs, at the end of 10 seconds represents the net area of the current curve with respect to the horizontal axis (areas above the horizontal axis are positive and those below are negative) from time $-\infty$ to 10 seconds. From minus infinity ($-\infty$) to -2 seconds, the curve area is zero; from -2 to 0 seconds, the area is 2 x 2 or 4 ampere-seconds (coulombs); from 0 to 3 seconds, the area is zero; from 3 to 7 seconds, the area is -4×1 or -4 ampere-seconds and from 8 to 10 seconds the area is $2 \times 4/2 = 4$ ampere-seconds. Therefore the net area from -2 to 10 seconds is $4 - 4 + 4 = 4$ ampere-seconds (coulombs) and the charge, q, at 10 seconds is 4 coulombs.

The second semester electrical course revolves around the sine wave. Electrical power is generated world wide in a sinusoidal form. In a resistor the voltage, $v_R = iR$; in the inductor, $v_L = L\,di/dt$ and in the capacitor, $v_C = \frac{1}{C}\int i\,dt$. Therefore, if the current generated is sinusoidal, then the voltage across each of the electrical elements will also be sinusoidal. This allows the voltage across the three elements to be compared as to phase since all have the same wave form, sinusoidal.

The concept of repeating or periodic waveforms is introduced. The alternating waveform (as a special case of the periodic waveform) is introduced where the definition of $f(\alpha) = -f(\alpha + \pi)$ holds. Then the concept of the sine wave (or cosine wave) as a special shape of the alternating waveform is finally introduced and studied in detail.

The sine wave is shown in the figure below as the variable current:



The abscissa is shown in degrees, radians and seconds.

The period, T, is the time in seconds to complete one cycle (a complete sine wave).

Since in the real world the current varies with time, the plot versus time is of considerable importance.

The frequency, f, in cycles per second, is by definition related to the period, T, as $f = \frac{1}{T}$.

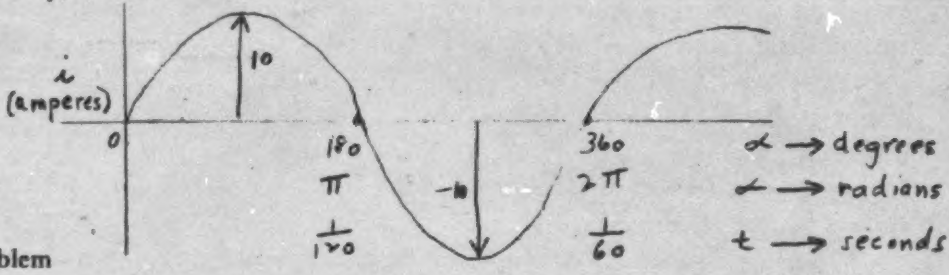
The amplitude or peak value and instantaneous values are discussed.

Problem

If $i = 10 \sin 377t = 10 \sin \alpha$, find the period, the frequency and sketch the sine wave versus the angle, α, in degrees and radians and also versus the time, t, in seconds.

The number 377 has the units of radians per second if t is in seconds, and may be expressed in a more convenient form as $2\pi\,(60)$. For $377t = \alpha$ to be 2π radians, then the value of $t = 1/60$ second (the period, T).

Therefore, the period, T, is 1/60 second and the frequency, f, is the reciprocal of T and is 60 cycles per second (cps or hertz). This is the standard frequency in the United States for the generation of electrical power. The term multiplying t, 377, is in radians per second and is called the angular velocity (related to revolutions per second). Obviously the angular velocity, $w = 2\pi f = 2\pi/T$.



Problem

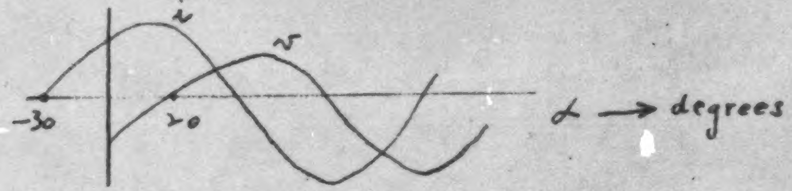
For the above waveform, if α = 30 degrees, what is the corresponding instantaneous value of time?

$$\frac{t}{T} = \frac{30}{360} \qquad t = \frac{1}{60} \times \frac{30}{360} = \frac{1}{720} \text{ second}$$

A very important concept is to compare two sine waves of the same frequency as to phase.

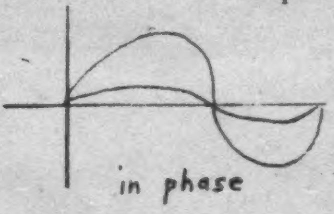
Problem

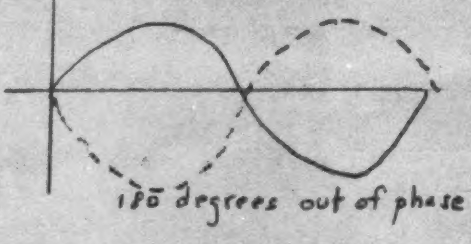
What is the phase relation between the sinusoidal current and voltage wave (of the same frequency) shown below?



The phase relation is independent of the amplitudes and is concerned with comparing identical points of two sine waves (the two positive peaks, the points where both are zero in magnitude and increasing positively). In the above figure, the electrical phraseology is that the current wave, i, leads (starts out earlier in time) the voltage wave, v, by 50 degrees. It may also be stated that the voltage wave lags the current wave by 50 degrees.

Two waveforms that are in phase or 180 degrees out of phase are shown below.



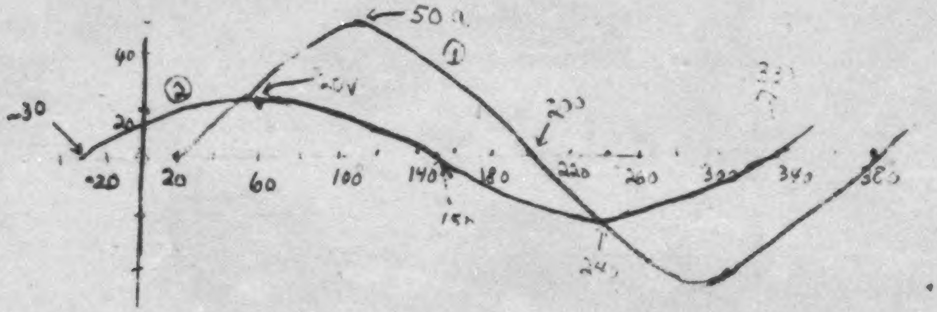


Sketching of sine waves is important.

Problem

Sketch the following waveforms: (1) $i = 50 \sin (\alpha - 20)$ (2) $v = 20 \sin (\alpha + 30)$

Waveform (1) starts 20 degrees to the right of the origin. Waveform (2) starts 30 degrees to the left of the origin.



Problems

What is the phase relation between the following waveforms?

(1) $i = 10 \sin (\alpha + 50)$ $v = 3 \sin (\alpha + 30)$
i leads v by 20 degrees or v lags i by 20 degrees

(2) $i = -10 \sin (\alpha + 120)$ $v = 5 \sin (\alpha - 60)$
The minus sign indicate a phase shift of ±180 degrees; therefore to remove the minus sign, shift the wave ±180 degrees.

$i = -10 \sin (\alpha + 120) = 10 \sin (\alpha - 60)$ $v = 5 \sin (\alpha - 60)$

The wave forms are in phase.

(3) $i = 3$ cosine $(\alpha - 20)$ $v = 72$ sine $(\alpha + 30)$

To change a cosine wave to a sine wave, simply add 90 degrees; that is, $\cos \alpha = \sin (\alpha + 90)$.

$i = 3 \cos (\alpha - 20) = 3 \sin (\alpha + 70)$ $v = 72 \sin (\alpha + 30)$

The current wave leads the voltage wave by 40 degrees or the voltage wave lags the current wave by 40 degrees.

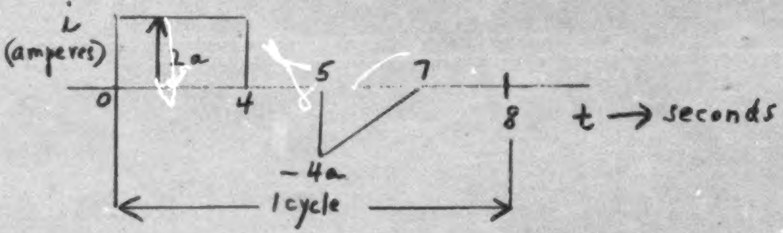
Two important values of wave forms (periodic) are the average and effective values over a complete cycle.

The average value is simply the net area divided by the length of the cycle of the waveforms.

Problems

Find the average value over the complete cycle for the following:

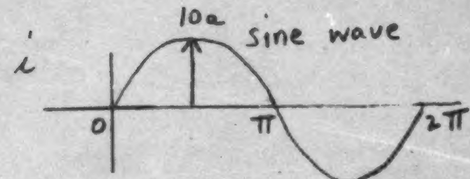
(1)



$$\text{The net area is } 2 \times 4 - \frac{4 \times 2}{2} = 8 - 4 = 4 \text{ amp-sec.}$$

$$I_{ave} = \frac{4 \text{ amp-sec}}{8 \text{ sec}} = 0.5 \text{ amperes.}$$

(2)



The average value is zero by inspection, since the area of the positive alternation is equal and opposite to that of the negative alternation. The average value of a sine wave is zero over the full cycle. In the laboratory, a dc ammeter, which reads the average value of a periodic waveform over one cycle, is used to measure sinusoidal current. It is therefore obvious (and one is not "shocked") that the reading will be zero.

(3) Of interest to the technician is the average value of one alternation. Referring to the previous problem and using the calculus:

$$I_{ave} = \frac{1}{\pi} \int_0^{\pi} 10 \sin \alpha d\alpha = \frac{10}{\pi} \left[-\cos \alpha\right]_0^{\pi}$$

$$I_{ave} = \frac{10}{\pi} [1 + 1] = \frac{2}{\pi} (10) = \frac{20}{\pi} \text{ amperes}$$

As a general rule, the average value for one alternation is $2/\pi$(or 0.636) times the peak or maximum value of the sine wave.

The student is also asked to graphically compute the above value by plotting the sine wave on a graph sheet which is "boxed" and obtaining the area of the alternation by counting the number of boxes included between it and the horizontal axis. The average value is then obtained by dividing by the number of boxes along the abscissa corresponding to the length of the alternation.

Of equal importance to the average value is the effective or RMS (root-mean-square) value of a repeating waveform since in electrical circuits this value is a measure of the amount of electrical power converted into heat.

The effective value is defined as:

$$I_{eff} = \sqrt{\frac{1}{2\pi} \int_{0}^{2\pi} [i(\alpha)]^2 \, d\alpha}$$

or

$$I_{eff} = \sqrt{\frac{1}{T} \int_{0}^{T} [i(t)]^2 \, dt}$$

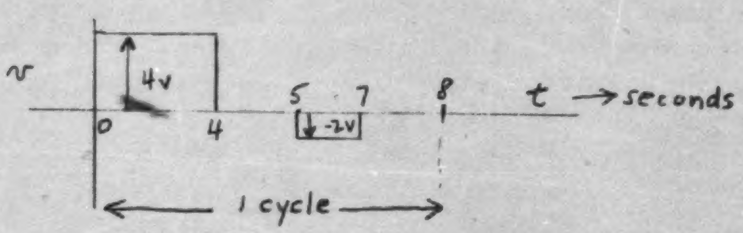
Since the student, at this point, has yet to be exposed to integration in his mathematics course, the following steps are outlined to him to obtain the answer graphically.

To obtain the effective value:

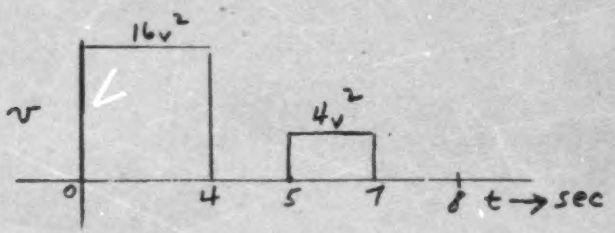
(1) Square each ordinate of the given waveform and plot this new "squared" waveform.
(2) Obtain the net area of the waveform generated in (1) over the complete cycle.
(3) Obtain an average value by dividing by the length of the period.
(4) Take the square root of the answer obtained in (3).

Problems

(1) Find the effective value of the repeating voltage waveform shown below over one cycle.



Step 1–Square each ordinate and sketch new wave.



Step 2

Obtain the net area over one cycle

Net area = 16 x 4 volt^2 - sec + 4 x 2 (note that all areas are always positive due to the squaring of the ordinate).

Net area = 72 volt^2- sec

Step 3

The average value of Step 2 over 1 cycle is $\dfrac{72\ V^2 - \text{sec}}{8\ \text{sec}} = 9\ \text{volt}^2$

Step 4

The effective value of voltage is now: $E_{eff} = \sqrt{9\ \text{Volt}^2} = 3$ volts

The practical significance of the 3 volts is that this constant value of voltage gives the same heating effect as the original repeating waveform of voltage.

(2) Find the effective value of the sine wave, $i = 10 \sin \alpha$, over one cycle.

The student is asked to do this problem graphically on boxed graph paper and obtain the area under the square curve by counting boxes. Following steps 1–4 the answer will be approximated 7.1 amperes.

Later in the semester as the student proceeds with calculus in his mathematics course, a general proof of the sine wave effective value is given

Let $i = 10 \sin \alpha$

$$\text{then } I_{eff} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi}[10 \sin \alpha]^2 \, d\alpha}$$

$$= \sqrt{\frac{100}{2\pi}\int_0^{2\pi}\sin^2 \alpha \, d\alpha}$$

but $\sin^2 \alpha = ½ - ½ \cos \alpha$

$$\therefore I_{eff} = \sqrt{\frac{100}{2\pi}\left[\int_0^{2\pi} ½ d\,\alpha - \int_0^{2\pi} ½ \cos \partial\alpha d\alpha\right]}$$

By inspection, the latter integral is zero, since the net area of a sinusoidal wave is zero over the full cycle.

$$\therefore I_{eff} = \sqrt{\frac{100}{2\pi}\int_0^{2\pi} ½ \, d\alpha}$$

$$= \sqrt{\frac{100}{2\pi}\left[\frac{\alpha}{2}\right]_0^{2\pi}}$$

$$I_{eff} = \sqrt{\frac{100}{2\pi}[\pi]} = \sqrt{\frac{100}{2}}$$

$$I_{eff} = \frac{10}{\sqrt{2}} = 0.707\ (10) = 7.07 \text{ amperes}$$

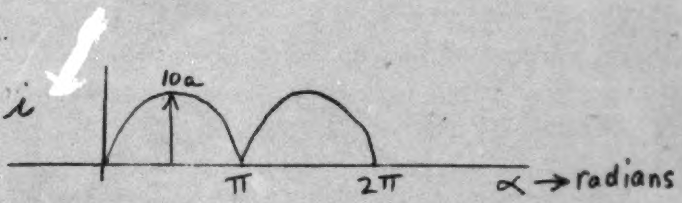
which approximates the answer of 7.1 amperes in problem (2).

In general, since 10 is the peak or maximum value, the effective value of a sine wave over one cycle is the maximum value over the square root of two or 0.707 times the maximum.

$$I_{eff} = \frac{I_{max}}{\sqrt{2}} = 0.707\ I_{max}$$

In lay language when a toaster (or any other appliance) is rated 7 amperes, it means that the effective value of the sine wave of current through the toaster is 7 amperes. The actual sine wave of current is $i = 7\sqrt{2} \sin \alpha$ and if the frequency is 60 cps, then the current is $i = 7\sqrt{2}$ sine 377 T in the time domain.

(3) Find the effective value of current for the full rectified sine wave shown below over one cycle.



The answer is immediately 10(0.707) = 7.07 amperes since it will be the same answer as for the "regular" sine wave (when the ordinate is squared, the new squared curve will be identical in both cases).

The student must become adept in the solution of electrical circuits with a sinusoidal input of voltage since this is the norm in "real" life. In linear circuits (those in which the electrical parameters, R, L and C are constant and do not depend on the input-cause), the effect (current) is always of the same shape as the cause (the input voltage). In non-linear circuits, the values of some of the circuit parameters vary with magnitude of the input voltage; thus the input (cause) and the output (effect) will differ in waveforms. This area of circuit theory is quite difficult and is treated in the more advanced electrical courses.

The solution of electrical linear circuits with sine wave inputs involves the addition, subtraction, multiplication, and division of sine waves of the same frequency. If this were to be done in the time domain, the mathematics would be cumbersome and time-consuming. The electrical engineer therefore developed a type of "electrical shorthand."

This electrical shorthand converts sinusoidal time expressions into the phaser (complex numbers) domain. The mathematics is performed much more easily in the phaser domain and then the answer is converted back, if necessary, into the time domain. However, electrical parlance most often occurs in the phaser domain so that the time domain conversion is often not necessary.

It can be shown that sine waves can be converted to an "unreal" world for expediency in mathematical solutions. For instance:

time domain		phaser domain
$i = 10 \sin (377\,t + 30^\circ)$	$\approx$	$10 \angle 30^\circ$
$v = 5 \cos (377\,t - 100^\circ) = 5 \sin (377\,t - 10^\circ)$	$\approx$	$5 \angle -10^\circ$

Note that conversion to the phaser domain does not show the frequency (60 cps in this case), but always assumes that in any mathematical solution, the sine waves are of the same frequency. The phaser only shows the peak value and the phase angle of the sine wave. The phase angle is referenced to the sine wave originating at the origin (sin α).

Since all "electrical talk" refers to effective values, the conversion to the phaser domain should always show effective values (unless otherwise stated); therefore, in electrical shorthand, the conversion is actually done as follows:

time domain		phaser domain
$i = 10 \sin (377\,t + 30)$	$\approx$	$10\,(.707) \angle 30 = 7.07 \angle 30$
$v = 5 \operatorname{cost} (377\,t - 100) = 5 \sin (377\,t - 10)$	$\approx$	$5\,(.707) \angle -10 = 3.54 \angle 10$

Problems

(1) Convert $i = 7 \sin(\alpha + 20)$ to the phaser domain $I = -7\,(.707)\angle 20 = -10.0\angle 20$

where I is the symbol for the phaser.

Note that the magnitude is expressed in effective values.

(2) $f = 25$ cps, $V = 10\angle -62^{\circ}$

Find the sinusoidal time expression for the voltage, v.

$f = 25$ cps $w = 2\pi\,(25) = 157$ radians/sec

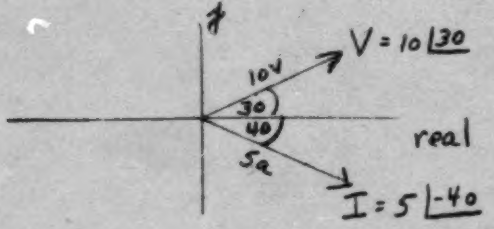
$$v = \frac{10}{0.707} \sin(157t - 62^{\circ}) = 14.14 \sin(157t - 62^{\circ})$$

Complex numbers are expressed in the polar form for ease of multiplication or division and in the rectangular or cartesian coordinate form for ease in addition and subtraction. The student must be able to convert quickly from one form to the other.

A slide rule session will show the ease of these conversions.

With the advent of the ubiquitous calculator, the ET Department is in the process of shifting from the slide rule to the calculator.

A phaser diagram with the abscissa as the real axis and the ordinate as the "j" axis ("i" axis is not used in order to avoid confusion with the electrical current, i) is used to show the phasers.



Phaser Diagram

The above phaser diagram shows that the voltage, V, leads the current, I, by 70 degrees or that I lags V by 70 degrees.

Problem

Convert 3 sin 157t + 4 cos 157t into one sinusoidal expression.

$$3 \sin 157t \approx 3\angle 0 = 3 + j0$$

$$4 \cos 157t \approx 4\angle 90 \quad \underline{0 + j4}$$

$$\text{Sum} = 3 + j4 = 5\angle 53$$

$$5\angle 53 \approx 5 \sin(157t + 53) \text{ Ans.}$$

Note, that the conversion to the phaser domain used peak values since a conversion back to the time was also necessary.

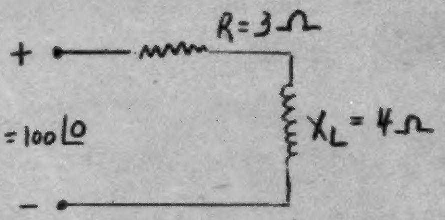
In the solution of sinusoidal electrical circuits the opposition (ohms) of the electrical elements must also be expressed as phasers (although it must be remembered that only voltages and currents are sinusoidal expressions of time) in order to have continuity of mathematics.

The resistor, R, has an impedance (opposition) of $R\angle 0$, the coil, L, has an impedance, X_L of $X_L\angle 90 = j^{X}L$ and the capacitor, C, has an impedance, X_C, of $X_C\angle -90 = -j^{X}C$.

Problem

Unless otherwise indicated, all phaser values of voltages and currents are effective values.

Solve the following series circuit for total impedance, total current, voltage across the resistor and the voltage across the coil. Draw the phaser diagram.



The total impedance, Z, = $3 + j4 = 5\angle 53$

$$I = \frac{V}{Z} = \frac{100\angle 0}{5\angle 53} = 20\angle -53$$

$$V_R = IR = 20\angle -53 \quad 3\angle 0 = 60\angle -53$$

$$V_L = IX_L = 20\angle -53 \quad 4\angle 90 = 80\angle 37$$

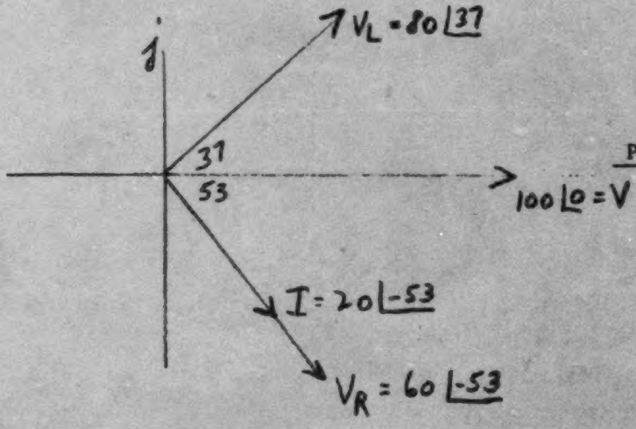
As a check using Kirchoff's voltage law, $V = V_R + V_L$

$$V_R = 60\angle -53 = 36 - j48$$

$$V_L = 80\angle 37 \quad = \underline{64 - j48}$$

$$\text{Sum} \quad = 100 + j0 \quad = 100\angle 0$$

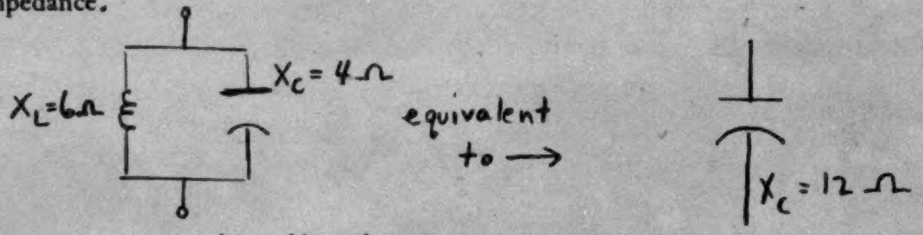
which checks with the value of the input voltage, V, that was given above.



Phaser Diagram

Problem

(1) Calculate the total impedance of the parallel circuit shown below and indicate the nature of this impedance.

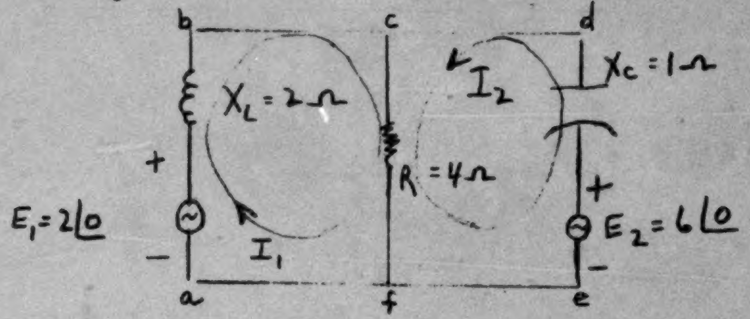


$$Z_T = \frac{\text{product of impedances}}{\text{sum of impedances}}$$

$$Z_T = \frac{6\underline{/90} \quad 4\,\underline{/-90}}{j6 - j4} = \frac{24\,\underline{/0}}{j2} = \frac{24\angle 0}{\underline{2\,/90}} = 12\angle{-90}$$

This represents a capacitive reactance, X_c, of 12 ohms.

(2) Find the current, I_1, in the circuit below using mesh analysis



loop abcfa (applying Kirchhoff's voltage law)

$$E_1 - I_1(X_L + R) - I_2R = 0$$

eq. 1 $\quad \therefore E_1 = I_1(R + X_L) + I_2R$

loop edcfe (applying Kirchhoff's voltage law)

$$E_2 - I_2(X_c + R) - I_1R = 0$$

eq. 2 $\quad E_2 = I_1R + I_2(R + X_c)$

substituting values into equation 1 yields:

eq. 1a $\quad 2\angle 0 = (4 + j2)\,I_1 + 4I_2$ $\qquad$ eq. 2a $\quad 6\angle 0 = 4I_1 + (4 - j1)\,I_2$

Solving for I, using determinants yields:

$$I_1 = \frac{\begin{vmatrix} 2\angle 0 & 4 \\ 6\angle 0 & 4 - j1 \end{vmatrix}}{\begin{vmatrix} 4 + j2 & 4 \\ 4 & 4 - j1 \end{vmatrix}} = \frac{2(4 - j1) - 4(6)}{(4 + j2)(4 - j1) - 4(4)}$$

$$I_1 = \frac{8 - j2 - 24}{16 + j4 + 2 - 16} = \frac{-16 - j2}{2 + j4} = \frac{16.1\,\underline{/-173}}{4.47\,\underline{/63}}$$

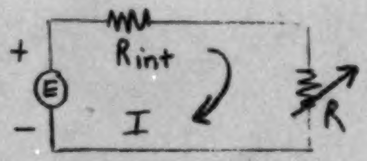
$$I_1 = 3.6\angle{-236} \quad \text{or} \quad 3.6\angle 124 \ \text{amperes}$$

An ac ammeter placed in the circuit would read 3.6 amperes (effective value).
If the frequency were 60 cps, then conversion to the time domain would yield:

$$i_1 = 3.6\sqrt{2}\,\sin(377t + 124)\ \text{amperes}$$

Problem

In the circuit below find the value of R for maximum power transfer to it and then compute the value of this maximum power.



E = 100 0 volts

$R_{int.}$ = 2

$$P = I^2R$$

The power, P, dissipated in the resistor, R, is equal to I^2R watts but $I = \dfrac{E}{R + R_{int}}$

$$P = \frac{E^2R}{(R + R_{int})^2}$$

$$\frac{dP}{dR} = \frac{(R + R_{int})^2\,E^2 - RE^2\,(2)\,(R + R_{int})}{(R + R_{int})^4}$$

To find the maximum value (slope = 0), set the derivative equal to zero.

$$0 = \frac{dP}{dR} = \frac{(R + R_{int})^2\,E^2 - RE^2\,(2)\,(R + R_{int})}{(R + R_{int})^4}$$

$$0 = (R + R_{int})^2 - 2\,R\,(R + R_{int})$$

$$0 = R^2 + 2R\,R_{int} + R_{int} - 2R^2 - 2R\,R_{int}$$

$$0 = R_{int}^{\;2} - R^2 = (R_{int} + R)\,(R_{int} - R)$$

$R_{int} + R = 0$ $R = -R_{int}$ (reject, since physically there is no negative resistor)

$R_{int} - R = 0$ $\underline{R = R_{int}}$

Since physically the minimum power would occur in R when R = 0, the value shown indicates the value for maximum power.

Therefore in a series circuit the maximum power occurs in a load resistor when its value is equal to the internal resistance, $R_{int.}$, of the source. This is an important theorem, known as the "Maximum Power Transfer Theorem".

In this problem therefore set $R = R_{int.} = 2$ ohms.

For the condition for maximum power transfer,

$$I = \frac{E}{R_{int} + R} = \frac{100}{2 + 2} = 25 \text{ amperes}$$

$$P_{max} = I^2R = (25)^2\,(2) = 1250 \text{ watts.}$$

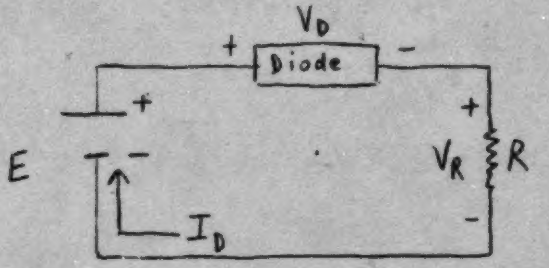
Although the power transferred to R is a maximum when $R = R_{int.}$, the efficiency of transmission is only 50 per cent since an equal amount is being dissipated in $R_{int.}$.

This problem is checked experimentally in the laboratory.

ELECTRONICS

In electronics a problem frequently encountered is the graphical solution of a straight line and a non-linear curve. The non-linear curve is usually one supplied by the manufacturer and may refer to the volt-ampere relations in a vacuum diode (two-element device) or in a semi-conductor diode.

Refer below to a fundamental diode circuit.



where E and R are given circuit values

Writing Kirchhoff's voltage law around the series loop yields: $E = V_D + I_DR$

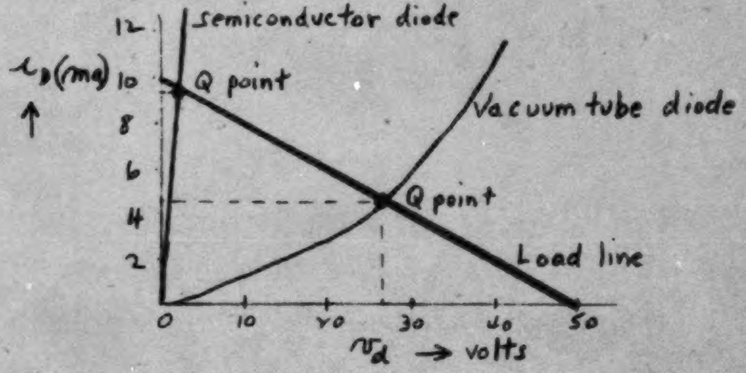
Solving for I_D gives: eq. 1 $I_D = \frac{-V_D}{R} + \frac{E}{R}$ where the variables are I_D (ordinate) and V_D (abscissa).

Equation 1 is in the form $y = mx + b$ where $y = I_D$, $m = -\frac{1}{R}$, $x = D$ and b is the ordinate (or y) intercept $= \frac{E}{R}$

The abscissa (or x) intercept may be found by setting $I_D(y)$ equal to zero. This yields the abscissa intercept as $V_D=E$.

Equation 1 has 2 variables (I_D and V_D) and, therefore, a minimum of 2 equations is necessary for their solution.

The second equation is supplied in graphical form by the manufacturer and is shown below for the two types of diode.



If the given circuit values are E = 50 volts and the load resistor, R, is 5 kilohms, then equation (1), often called the load line, is a straight line with the ordinate intercept equal to

$$\frac{E}{R} = \frac{50v}{5k\Omega} = 10 ma$$

and the abscissa intercept equal to E = 50 volts. Having these two points on the straight line, the straight line is plotted above. For the vacuum tube diode, the Q or operating point is obtained as the intersection of this straight line with the manufacturer's characteristics. The Q point yields approximate values of $V_D = 27$ volts and $I_D = 4.9$ ma.

In a similar fashion, for the semiconductor diode, the Q point coordinates are:

$V_D = 3$ volts and $I_D = 9.3$ ma.

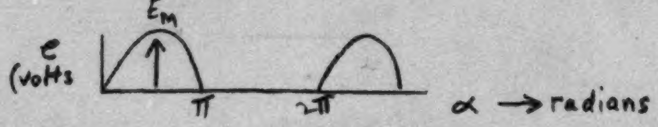
Problem

(1) Using Shockley's equation, $I_D = I_{DSS}\left(1 - \frac{V_{GS}}{V_P}\right)^2$, find the transistor current, I_D,

if $I_{DSS} = 10$ma and $V_P = -5$v (manufacturer's data) and $V_{GS} = -2.5$ v.

$$I_D = 10\left(1 - \frac{-2.5}{-5}\right)^2 = 10\,(½)^2 = 2.5\text{ma}$$

(2) For the half wave rectified sine wave shown, compute the average value over the full cycle.



$$E_{AV} = \frac{1}{2\pi}\int_0^{2\pi} v(\alpha)\ d\alpha = \frac{1}{2\pi}\int_0^{\pi} E_M \sin\alpha\ d\alpha$$

$$E_{AV} = \frac{E_M}{2\pi}\,[-\cos\alpha]_0^{\pi} = \frac{E_M}{2\pi}\,[1-(-1)] = \frac{E_M}{\pi} = 0.318\,E_M$$

In general, the average value of a half wave rectified sine wave over the full cycle is 0.318 times the maximum value.

The decibel (db) had as its origin the fact that power and audio levels are related on a logarithmic basis. When referring to voltage levels, the decibel is defined by:

$$\text{db} = 20 \log_{10} \frac{V_2}{V_1}$$

Problems

(1) If V_2/V_1 is 100, how many decibels does it represent?

$$\text{db} = 20 \log_{10} 100 = 20\,(2) = 40 \text{ db.}$$

(2) If the decibels are equal to 6, what is the voltage ratio, V_2/V_1?

$$6 = 20 \log_{10} N \quad \text{when} \quad N = \frac{V_2}{V_1}$$

$$\log_{10} N = \frac{6}{20} = 0.30$$

$$N \simeq 2.0 = \frac{V_2}{V_1}$$

In electrical terms, if the voltage gain, V_2/V_1, drops by 6 decibels, it means that the ratio, V_2/V_1, is down by a factor of 2 or the new gain is ½ of the former voltage gain. If the voltage gain increases by 6 decibels, it signifies that the voltage gain has doubled over the previous value.

The significance of 0 db indicates that the voltage ratio is unity.

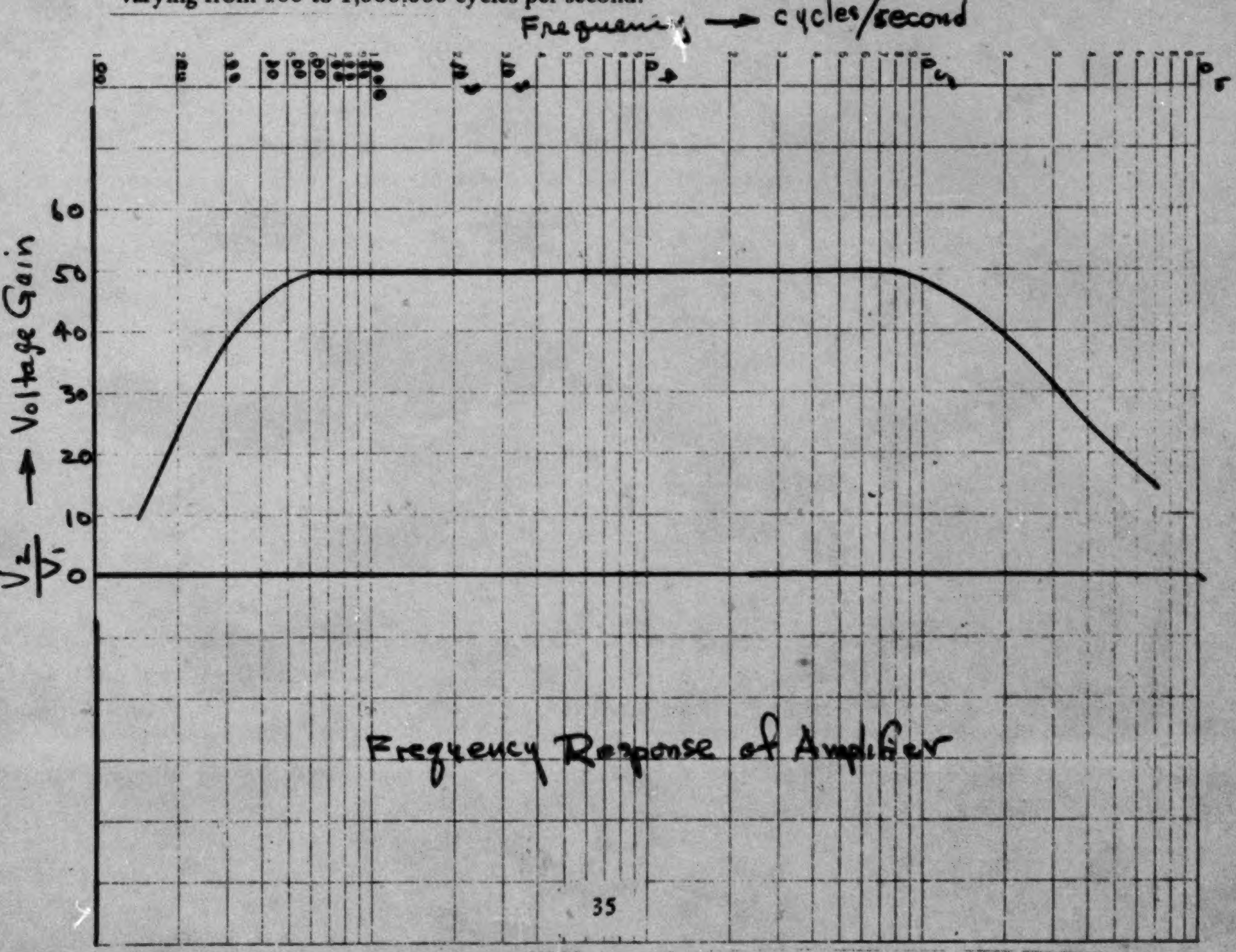
$$\text{db} = 20 \log_{10} \frac{V_2}{V_1} \qquad 0 = 20 \log_{10} \frac{V_2}{V_1} \qquad 0 = \log_{10} \frac{V_2}{V_1}$$

$$\therefore \frac{V_2}{V_1} = 1$$

Or if one remembers that the log of a number is the exponent to which the base must be raised to equal the number, it follows that

$$10^0 = \frac{V_2}{V_2} = 1 \quad \text{since 10 to the zero exponent is unity}$$

The frequency response of an amplifier is a plot of the voltage gain, V_2/V_1, (as a dimensionless ratio) versus frequency. For convenience, since the range of frequencies is very large, this plot is done on semi-log paper. It should be noted that no zero frequency can ever appear on the abscissa (see the attached semi-log sheet, which is a 4-cycle paper with the frequency varying from 100 to 1,000,000 cycles per second.



TRANSIENT CIRCUIT ANALYSIS

Historically, the student in electrical technology is exposed to a higher level of mathematics than those in other engineering technologies. The ET student may be required to take a math course beyond the applied technical calculus course which would involve an introduction to differential equations and the Laplace transform. This additional mathematics is essential to solve problems involving the transient conditions in an electrical circuit.

For example, in turning on a switch to light a fluorescent bulb, it takes a finite amount of time for the bulb to light after the switch is in the "on" position. The condition after the bulb is finally lighted is termed the steady state or final condition. What transpires before the steady state condition is reached would come under the area of technology known as the transient condition.

In some colleges, the course is taught by the mathematics department. At Queensborough, the course, ET-13. Transient Circuit Analysis, is taught by the technical department. This appears to be more logical since the practical applications of the mathematics are of paramount interest to the technical student.

Undoubtedly, one of the most important mathematical functions in all of pure and applied science and engineering, including transients in electrical circuits, is that general function, Ae^{st}, which duplicates its shape when differentiated or integrated. Let us first consider the function when "s" is equal to a negative real number.

As an example, consider the function, $i = e^{-2t}$, where "s" is evidently equal to -2. It is desired to plot this function versus "t" from the engineering viewpoint.

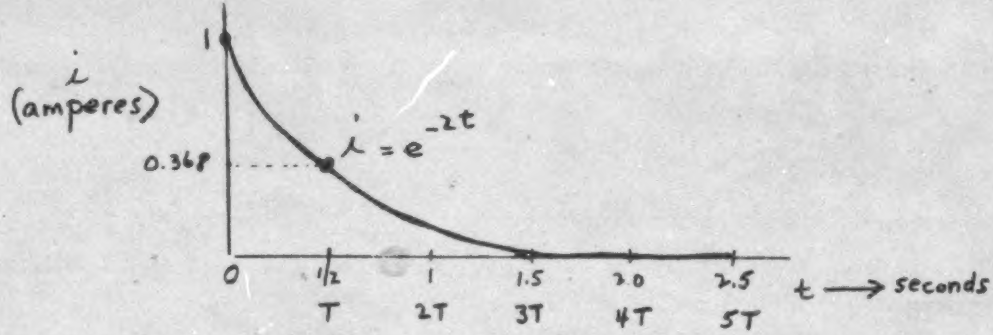
A time equal to T is defined as the time constant. This time constant is defined as the value of "t", which makes the exponent of the function equal to one in magnitude. From this definition, the time constant, T, is equal to ½ second.

The table shown below is filled in for values of t from 0 to five time constants in steps of one time constant.

t	e^{-2t}		
0	1	= 1	= 1
½ (T)	e^{-1}	= 0.368	= 0.368
1 (2T)	e^{-2}	$= (0.368)^2$	= 0.135
$\frac{3}{2}$ (3T)	e^{-3}	$= (0.368)^3$	= 0.0498
2 (4T)	e^{-4}	$= (0.368)^4$	= 0.0183
$\frac{5}{2}$ (5T)	e^{-5}	$= (0.368)^5$	= 0.00674

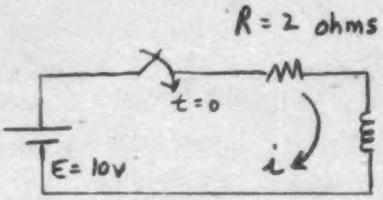
The above table indicates that, in five time constants, the value of the original function at t = 0 has decreased to a value equal to 0.674 per cent of the initial value (t = 0). For practical purposes in engineering, it is said that the function has dropped to a "zero" value (that is, to a value much smaller than its initial value). After five time constants have elapsed, the transient period is considered over or the transient value is now zero. The table also indicates that for each time constant, T, the function decreases by the factor 0.368.

The plot of $i = e^{-2t}$ for five time constants is shown below.



The course will be concerned with the solution of differential equations up to the second order which are linear and have constant coefficients.

Consider the circuit shown below.



Solve for i (t), for $t \geqslant 0_+$ and plot the curve.

From electrical theory, the differential equation (D.E.) is:

$$E = Ri + L\frac{di}{dt} \quad \text{(1st order)}$$

$$10 = 2i + 5\frac{di}{dt} \quad \text{(eq. 1)}$$

The complete solution for the current, i, consists of two parts – the forced response, i_f, due to the source (10 volts) and the natural response, i_n, due to the circuit configuration. The natural response is independent of the waveform or nature of the source.

The complete solution is the sum of these two components.

$i = i_f + i_n$

In mathematics the forced response is called the particular solution and the natural response is familiarly known as the complementary solution.

Since the forced response must have the same waveform as the input (which is the constant, 10) let $i_f = I$, a constant. Substituting into equation 1, yields

$10 = 2I + 5(0)$ since the derivative of a constant is zero

$10 = 2I$ or $I = i_f = 5$ amperes

The natural response is obtained by setting the source equal to zero. Therefore, equation 1 yields:

$$0 = 2i_n + 5\frac{di_n}{dt} \quad \text{(eq. 2)}$$

Equation 2 indicates that the solution for i_n (other than the trivial solution of $i_n = 0$) must be such that a function and its derivative must add up to zero for all values of t. This clearly indicates that the function and its derivative must have the same waveshape and one must be the negative of the other. Therefore, the solution for i_n is assumed to be:

$i_n = Ae^{st}$, where "s" will be a negative real number.

Substituting this solution into equation 2 yields:

$$0 = 2Ae^{st} + 5sAe^{st} \qquad 0 = 2 + 5s \quad \text{(eq. 3)} \qquad s = -\frac{2}{5}$$

Therefore, $i_n = Ae^{\frac{-2t}{5}}$ (eq. 4)

and the complete solution for the current is: $i = i_f + i_n = 5 + Ae^{\frac{-2t}{5}}$ (eq. 4)

Equation 3 is called the characteristic equation and is determined solely by the circuit elements and not by the waveform of the source.

There now remains the computation for the constant, A, to complete the solution.

Using electrical theory at $t = 0_+$, $i = 0$. Substituting this boundary condition into equation 4 yields:

$$0 = 5 + Ae^{\frac{-2(0)}{5}} = 5 + A \qquad \text{or} \qquad A = -5$$

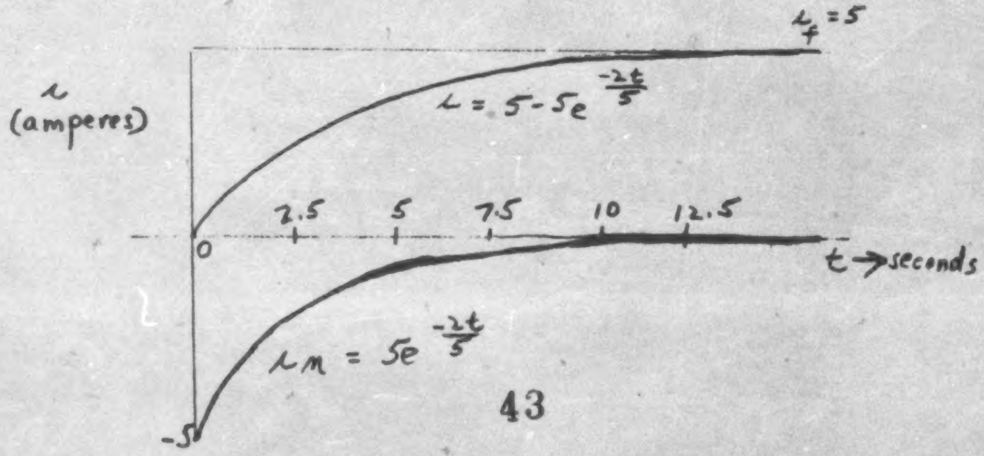
$$i = 5 - 5e^{\frac{-2t}{5}} \quad (\text{for } t \geqslant 0_+)$$

The first term, 5, represents the forced response, and the second term, $-5e^{\frac{-2t}{5}}$ represents the natural (or transient) response. It is to be noted that, in practical problems, the transient response will approach zero as t approaches infinity. In engineering problems, the natural (transient) response will be zero in five time constants as indicated previously.

The time constant for the exponential term is: $T = \frac{5}{2} = 2.5$ seconds. Therefore, the natural response will die out (approach zero) in 5T or 12.5 seconds and the forced response (steady state value) will be a constant of 5 amperes analogous to the constant input of 10 volts.

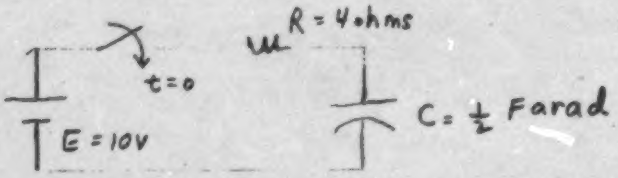
The characteristic equation (eq. 3) can be quickly obtained from equation 2 by the "mechanical" substitution of 1 for the variable, i_n, and s for the derivative of the variable, $\frac{di_n}{dt}$. If an integral term appears in equation 2, then the "mechanical" substitution can be shown to be 1/s.

The plot of the solution for the total current, i, is shown below as the sum of the constant, 5, and the exponential term: $-5e^{\frac{-2t}{5}}$



Example 1

For the circuit below, solve for the charge, q, for $t \geq 0_+$ and sketch the curve.



$$E = R\frac{dq}{dt} + \frac{q}{C}$$

$$10 = 4\frac{dq}{dt} + 2q$$

$0 = 4s + 2(1)$ characteristic equation

$0 = 4s + 2$

$s = -½$

$q_n = Ae^{\frac{-t}{2}}$ $\quad$ T = 2 seconds

$q_f = Q$ (constant)

$10 = 4(0) + \frac{Q}{½}$ or $Q = q_f = 5$ coulombs

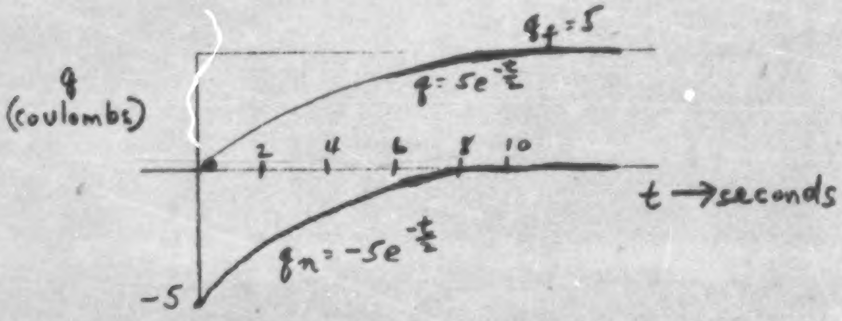
$q = q_f + q_n = 5 + Ae^{\frac{-t}{2}}$

at $t = 0_+$, $q = 0$

$0 = 5 + Ae^{\frac{-0}{2}} = 5 + A$

$A = -5$

$q = 5 - 5e^{\frac{-t}{2}}$ $\quad$ solution



Example 2

A circuit is defined by the equation shown below. How long will it take for the circuit to reach the steady state response? (Or how long will it take for the natural [transient] response to die out?)

$$3 \sin 5t = 2i + 3\int i\,dt$$

The equation for the complementary solution is obtained by setting the source equal to zero.

$$0 = 2i + 3\int i\,dt$$

The characteristic equation is determined by making the appropriate mechanical substitutions.

$$0 = 2\,(1) + 3\left(\frac{1}{s}\right) \qquad -2 = \frac{3}{s} \text{ or } s = -\frac{3}{2} \qquad i_n = Ae^{\frac{-3t}{2}}$$

The time constant is therefore 2/3 seconds.

The natural response will die out in 5 time constants, or 10/3 seconds.

Example 3

A circuit is defined by the equation shown below. Find the forced response only.

$$100 \sin 10\,t = 3\frac{dv}{dt} + 40v$$

The forced response, v_f, must be a sine wave of the same form as the input, 100 sin 10t.

$v_f = K \sin (10t + \theta)$, or it may be expressed in the more convenient form,

$$v_f = A \cos 10\,t + B \sin 10t$$

Substitution into the differential equation yields:

$$\begin{aligned} 100 \sin 10t &= 3\,(-10\,A \sin 10t + 10B \cos 10t) \\ &\quad + 40A \cos 10t + 40B \sin 10t \\ 100 \sin 10t &= (-30A + 40\,B) \sin 10t \\ &\quad + (30B + 40A) \cos 10t \end{aligned}$$

Equating the coefficients of the sine and cosine terms on both sides of the equation yields:

$$100 = -30A + 40B \text{ or } 10 = 3A + 4B \qquad 0 = 40A + 30B \text{ or } 0 = 4A + 3B$$

Solving for A and B yields: $A = -\frac{6}{5}$, $B = \frac{8}{5}$

Therefore, the solution for the forced response is: $v_f = \frac{8}{5} \sin 10t - \frac{6}{5} \cos 10t$

The sine and cosine terms can be combined into one sinusoidal term by using phasers.

$$\frac{8}{5} \sin 10t \approx \frac{8}{5}\angle 0 = \frac{8}{5} \qquad -\frac{6}{5} \cos 10t \approx -\frac{6}{5}\angle 90 = -j\,\frac{6}{5}$$

$$\frac{8}{5} - j\,\frac{6}{5} = \frac{10}{5}\angle -37 = 2\angle -37$$

The conversion back from the phaser domain into the time domain yields:

$$v_f = 2 \sin (10t - 37^\circ)$$

Example 4

Solve the following differential equation for i (t), for $t \geq 0_+$, and sketch the curve. The initial conditions are: $i(0_+) = 0$ and $di/dt\,(0_+) = 0$

$$\frac{d^2i}{dt^2} + 4\frac{di}{dt} + 3i = 270 \quad \text{(eq. 5)}$$

Let $i_f = I$ and substituting into eq. 5

$0 + 4(0) + 3I = 270 \qquad i_f = I = 90$

Let the natural response be $i_n = Ae^{st}$ and obtaining the characteristic equation by mechanical substitution (remember that the substitution for the second derivative is s^2).

$s^2 + 4s + 3(1) = 0$

$s^2 + 4s + 3 = 0$

$(s + 3)\ (s + 1) = 0 \qquad s_1 = -3, \quad s_2 = -1 \qquad i_n = A_1e^{-t} + A_2e^{-3t}$

There are now two time constants. One is equal to 1 second and the second is equal to 1/3 second. The first term dies out in 5 seconds and the second in 5/3 seconds. Therefore, using the longer time constant, the natural response will die out (approach zero) in 5 seconds.

$i = i_f + i_n = 90 + A_1e^{-t} + A_2e^{-3t}$ (eq. 6)

at $t = 0_+$, $i = 0$ therefore from eq. 6 $\qquad 0 = 90 + A_1 + A_2$ (eq. 7)

at $t = 0_+$, $\frac{di}{dt} = 0$

differentiating equation 6 yields: $\qquad \frac{di}{dt} = 0 - A_1e^{-t} - 3A_2e^{-3t}$ (eq. 8)

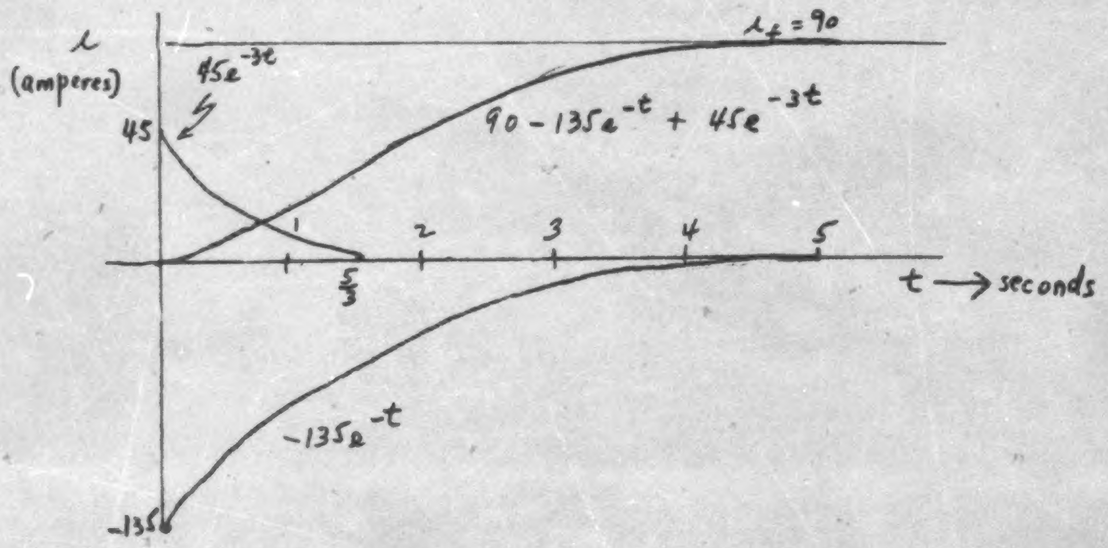
substituting the boundary condition into eq. 8

$0 = 0 - A_1 - 3A_2 = - A_1 - 3A_2$ (eq. 9)

Solving equations 7 and 9 simultaneously yields:

$A_1 = -135 \qquad A_2 = 45 \qquad i = 90 - 135e^{-t} + 45e^{-3t}$ (solution)

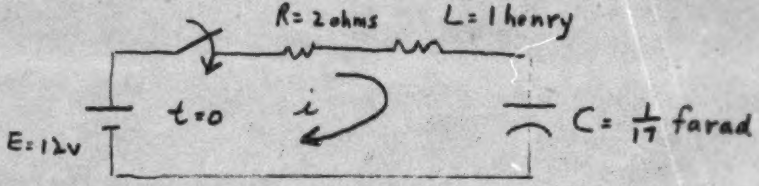
The plot of the curve is shown below.



The above plot for the total current, i, is known as the non-oscillatory or overdamped case. This occurs in the second order system when the values of s are real, negative and unequal.

Example 5

Solve the following circuit for i(t), for t $\geqslant 0_+$ and sketch the curve.



$$E = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt$$

$$12 = 2i + 1\frac{di}{dt} + 17\int idt \quad \text{(eq. 10)}$$

From electrical theory, the forced response of the current, i_f, is zero. The boundary conditions at $t = 0_+$ are $i = 0$ and $di/dt = 12$ amperes per second.

The characteristic equation from equation 10 is

$$0 = 2\,(1) + 1s + 17\left(\frac{1}{s}\right)$$

$$0 = 2 + s + \frac{17}{s}$$

$$0 = 2s + s^2 + 17$$

$$0 = s^2 + 2s + 17 \qquad s = \frac{-2 \pm \sqrt{4-68}}{2} = -1 \pm j4$$

The natural response, i_n, is of the form Ke^{st}

$$i_n = K_1e^{(-1+j4)t} + K_2e^{(-1-j4)t}$$

$$i_n = e^{-t}\,(K_1e^{j4t} + K_2e^{-j4t}) \quad \text{(eq. 11)}$$

Recalling that $\frac{e^{jx} + e^{-jx}}{2}$ is the sinusoidal wave, cos x, then equation 11 may be written as $i_n = e^{-t}\,(K_3 \cos 4t + \theta)$

or in the more convenient form of:

$$i_n = e^{-t}\,(A \sin 4t + B \cos 4t)$$

$$i = i_f + i_n = 0 + e^{-t}\,(A \sin 4t + B \cos 4t) \qquad i = e^{-t}\,(A \sin 4t + B \cos 4t) \quad \text{(eq. 1}$$

at $t = 0_+$, $i = 0$

Therefore, equation 12 yields: $0 = 1\,[A\,(0) + B\,(1)]$ $\qquad \underline{B = 0}$

Differentiating equation 12 yields:

$$\frac{di}{dt} = e^{-t}\,(4A \cos t - 4B \sin 4t)$$

$$+ (A \sin 4t + B \operatorname{cost} 4t)\,(-1)\,(e^{-t})$$

at $t = 0_+$, $\frac{di}{dt} = 12$

$$12 = 1\,[4A\,(1) - 4B\,(0)] + [A(0) + B(1)]\,(-1)\,(1) \qquad \text{Since } B = 0$$

$$12 = 4A \qquad A = 3$$

Therefore, the solution for the total current, i, is:

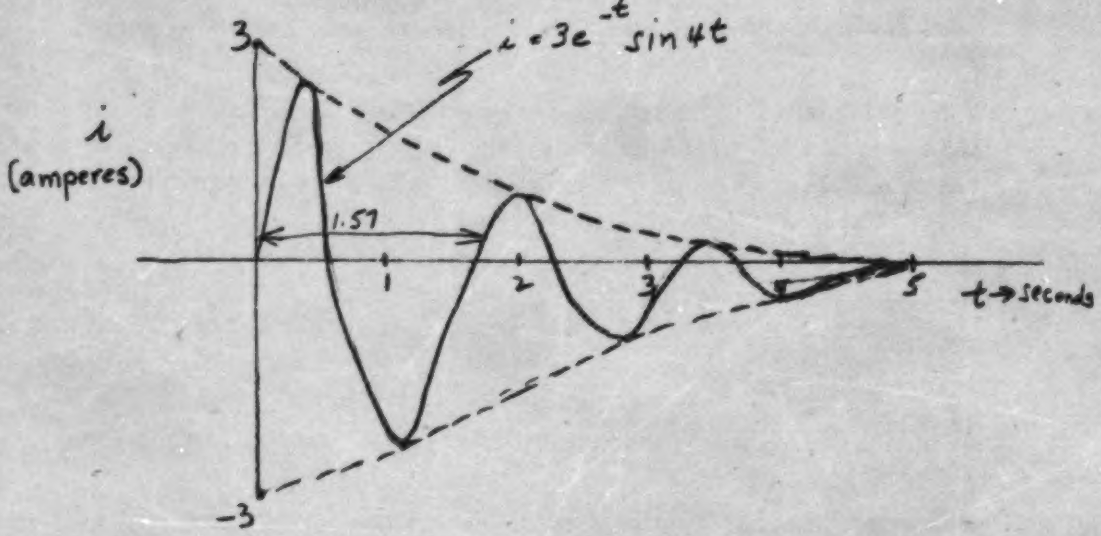
$\underline{i = 3e^{-t} \sin 4t}$ (Solution)

The above solution is called the damped sine wave. The time constant of the exponential term is 1 second and it will die out in five time constants or 5 seconds. The period (time for one cycle) of the sine wave is:

$$\frac{2\pi}{4} = \frac{\pi}{2} = 1.57 \text{ seconds.}$$

Since the current will die out (approach zero) in 5 seconds, the sine wave will go through $\frac{5}{1.57}$ or 3.2 cycles before approaching zero.

The plot of this oscillatory or underdamped case is shown below. The sine wave oscillates between the envelope determined by $3e^{-t}$ and its mirror image $-3e^{-t}$.



Damped Sinusoidal Wave

The method previously outlined for solving electrical problems in the transient condition is known as the "classical" method. Electrical engineering technology has developed several other methods — easier and at times more sophisticated. These methods have as their basis nothing more than simple algebra.

As indicated previously, the signal of probably greatest importance in electrical engineering is the one indicated below:

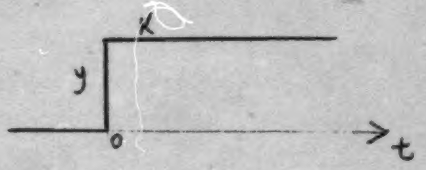
$$y = Ke^{st} \qquad \text{(eq. 1)}$$

The nature of the signal in equation 1 is a function of the amplitude, K, which is simply the value of the function, y, at t = 0 and of the quantity, s, which is called the complex frequency. It will be shown that the function, y, will assume a variety of waveforms depending upon the value of the quantity. s. In fact, by the superposition principles of summing exponential terms having different values of K and s, almost any wave shape of engineering importance can be generated.

If s = 0 is substituted into equation 1, the resultant function becomes:

$$y = Ke^{ot} = K \qquad \text{(eq. 2)}$$

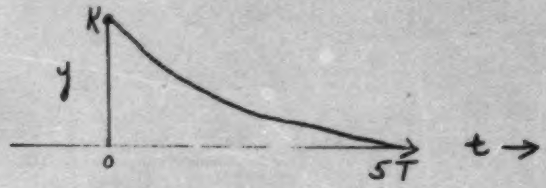
which is simply a constant (or direct current value of K. Since it is assumed that y = 0 for $t<0$ and y = K for $t>0$, the resulting "step" is shown below.



If s equals —"a" (when "a" is positive and real), then equation 1 becomes:

$$y = Ke^{-at} \qquad \text{(eq. 3)}$$

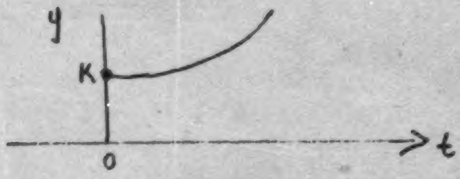
This is the waveform of the decreasing exponential with a time constant, T, of 1/a seconds and which will decay to zero for practical engineering purposes in 5 time constants. This waveform is shown below and a thorough discussion of its plot was given previously.



If the value of s is positive and real and equal to "a", then equation 1 becomes:

$$y = Ke^{at} \qquad \text{(eq. 4)}$$

This waveform is an increasing exponential curve which increases to infinity with time. This function is of negligible practical interest from an engineering point of view. It is shown below.



For the value of s pure imaginary and equal to jw, then equation 1 becomes:

$$y = Ke^{jwt}$$

but since imaginary values occur in complex conjugate pairs, equation 1 would yield:

$$y = K\,(e^{jwt} + e^{-jwt}) \quad \text{or the sinusoid}$$

$$y = A \text{ cosine } wt \qquad \text{(eq. 5)}$$

The plot of this waveform is obvious.

For s equal to a complex number with a negative real part, then equation 1 becomes:

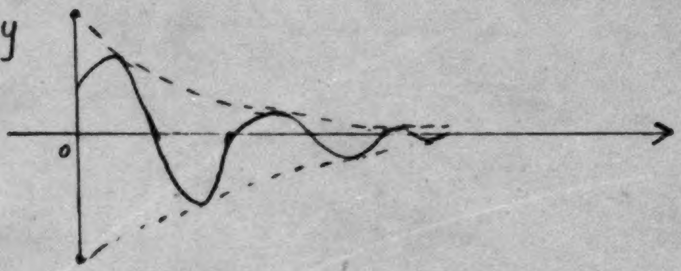
$$y = K(e^{-a + jwt} + e^{-a - jwt})$$

$$y = Ke^{-at}\,(e^{jwt} + e^{-jwt})$$

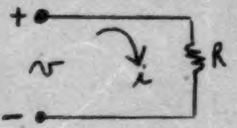
which is the exponentially damped sinusoid of the general form:

$$y = Ae^{-at}\ \text{sine}\,(wt + \theta) \qquad \text{(eq. 6)}$$

The plot of this waveform was previously discussed and is sketched below.



If the voltage input is of the form $v = Ve^{st}$ or the current, $i = Ie^{st}$, the operational values of the opposition (impedance) may be easily derived for the three electrical components, R, L and C.



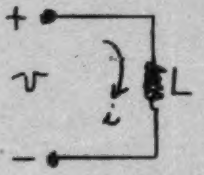
if $i = Ie^{st}$

then $v = Ri = RIe^{st}$

but $Z = \frac{V}{I}$ ohms (opposition)

however $V = RI$ (the amplitude of e)

so $Z = \frac{RI}{I} = R$ or the opposition (impedance) of a resistor, R, is the value of the resistance and independent of the value of s.



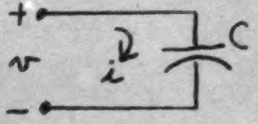
For the inductance, L,

$$v = L\,\frac{di}{dt}$$

if $i = Ie^{st}$ then $\frac{di}{dt} = sIe^{st}$

and $ve = LsIe^{st}$ where the amplitude of $v = LsI$

$$Z = \frac{V}{I} = \frac{sLI}{I} = sL,$$ or the opposition (impedance) of a coil is sL ohms and depends on the value of s.



$$i = C\,\frac{dv}{dt}$$

if $v = Ve^{st}$ then $\frac{dv}{dt} = Vse^{st}$

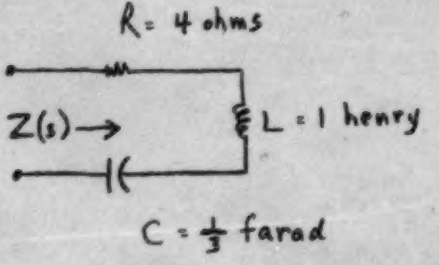
and $i = CVse^{st}$ where the amplitude of i is sCV.

$$Z = \frac{V}{I} = \frac{V}{sCV} = \frac{1}{sC}\ \text{ohms}$$

The impedance of a capacitor is inversely proportional to the value of s.

The above relationships are of importance in obtaining the characteristic equation of an electrical circuit without having to write the differential equation. If the value of the circuit impedance is set to zero, then the characteristic equation is easily obtained.

Example 1



In a series circuit, the toal impedance is the sum of the individual impedances.

$$\text{Therefore, } Z(s) = R + sL + \frac{1}{sC} = 4 + 1s + \frac{1}{s\frac{1}{3}}$$

Setting Z (s) = 0 yields the characteristic equation of the circuit:

$$0 = 4 + s + \frac{3}{s}$$

$$0 = s^2 + 4s + 3$$

$$(s + 3)(s + 1) = 0$$

$$s = -1, -3$$

Therefore the form of the natural (complementary) solution is:

$$Ae^{-t} + Be^{-3t}$$

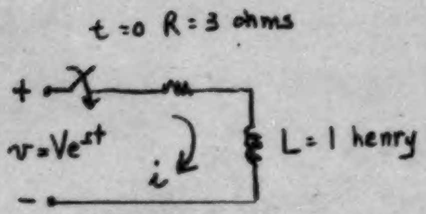
The forced (steady state or particular) solution is obtained from electrical theory. The complete solution (needing boundary conditions) is obtained from knowing the electrical input.

As the circuits become more complicated, simultaneous differential equations would have to be solved. The operational impedance method is relatively simple and requires only algebra.

The "transfer function," which is defined as the ratio of an output (effect) amplitude to the input (cause) amplitude for an exponential input of the form e^{st}, provides the complete solution to a transient problem. The transfer function will be denoted by T(s) and for a particular value of s, it represents the gain - the ratio of an output to an input.

In this method the substitution of the value of s corresponding to the input will yield the forced (particular) solution and setting the denominator equal to zero will yield the characteristic equation and thus the form of the natural (complementary) solution.

Example 2



Let v equal a dc (constant) input of 30 volts.

From simple electrical theory, the transfer function, T(s) is simply a ratio of the output (current) amplitude to the input (voltage) amplitude.

$$T(s) = \frac{I}{V} = \frac{1}{R + sL} = \frac{1}{3 + 1s} \qquad \text{(eq. 7)}$$

where R + sL is simply the impedance of the circuit and by Ohm's law,

$$V = IZ \text{ or } \frac{I}{V} = \frac{1}{Z} = \frac{1}{R + sL}$$

The input voltage has a magnitude of 30, and the value of s (for the input) is zero since $v = 30\,e^{st}$, and for dc, $s = 0$.

Substituting these values of the input into equation 7 yields the forced response for the current as follows:

$$\frac{I}{30} = \frac{1}{3 + 1\,(0)} = \frac{1}{3} \qquad i_f = I = \frac{30}{3} = 10 \text{ amperes (dc or constant)}$$

It is to be remembered that the forced response will always have the same waveform (shape) as the input (forcing function or cause).

Setting the denominator of equation 7 equal to zero will yield the characteristic equation.

$$3 + 1s = 0$$
$$s = -3$$

Therefore the form of the natural (complementary) solution is

$$i_n = Ae^{st} = Ae^{-3t}$$

The total solution is $i_f + i_n = i$

$$i = 10 + Ae^{-3t}$$

From simple electrical theory at $t = 0_+$ $i = 0$

$$0 = 10\,Ae^{-3(0)} = 10 + A$$
$$A = -10$$

The complete solution is therefore: $i = 10 - 10e^{-3t}$.

Example 3

Repeat example 2, if the input, v, is 40 sine 4t.

For the forced response of a sine wave $s = j4$ and the amplitude of the voltage is 20

$$\frac{I}{V} = \frac{1}{R + sL} = \frac{1}{3 + j4(1)}$$

$$\frac{I}{40} = \frac{1}{5\angle 53} \quad \text{or } I = \frac{40}{5\angle 53} = 8\angle{-53}$$

The above represents the current in the phasor form and must be converted back into the time domain.

$$i_f = 8 \text{ sine } (4t - 53)$$

Again, it must be remembered that the forced response will have the same waveform (same frequency of the sine wave) as the forcing function which is the sinusoidal voltage.

The natural response is independent of the input and is therefore the same as before (obtained by setting the denominator (R+sL or 3+1s) equal to zero.

$$i_n = AE^{-3t}$$

$$i = i_f + i_n = 8 \text{ sine } (4t - 53) + Ae^{-3t}$$

at $t = 0_+$, $i = 0$ as before

$$0 = 4 \sin(-53) + A$$

$$0 = 8(-.8) + A \text{ or } A = 6.4$$

The total solution for i is therefore

$$i = 8 \text{ sine } (4t - 53) + 6.4e^{-3t}$$

In the third method of the solution of differential equations, the student is introduced to the Laplace Transformation. (The first two methods are the classical and the transfer function.) In exposing the student to three methods, there is a flexibility in the solution of circuit problems. One method may have advantages over another. The classical method is the fundamental or basic one, and the other two have certain limitations.

The Laplace method converts (transforms) linear differential equations with constant coefficients into algebraic ones, thus greatly simplifying the steps leading to the solution.

The idea of a transform is certainly not new to the student. Logarithms transform multiplication into addition. That is, one went from one type of mathematical operation into a simpler one. The transformation was from a number domain into an exponential domain (logarithms). Work was then performed in this transformed or new domain (using log tables) and then the result or answer was obtained in the original domain using an inverse transformation (anti-logs).

Another example of transformation (mentioned previously) involved the solution of electrical circuits to a sinusoidal input. The sinusoidal time function (time domain) was transformed into complex algebra (phasor) domain where the mathematics involved is much simpler. The solution in the phasor domain was later transformed back into the sinusoidal time function (time domain).

The Laplace transform converts from the time (t) domain into the frequency (s) domain where the mathematical manipulations are much simpler and use can be made of Laplace tables. The inverse Laplace transform involves returning from the complex frequency domain into the time domain.

The Laplace method has distinct advantages over the classical method in the complete solution (forced and natural) of system networks. These include:

1. Reduction of linear differential equations with constant coefficients to linear algebraic equations.
2. Method is mechanical and straightforward.
3. Initial (boundary) conditions are immediately included in the transformed equations as a first rather than last step operation.
4. Time saving tables are available to speed up the time of solution.

It is pointed out, however, that the Laplace transform method solves no problem that can't be handled by the classical method. Although it has the advantages indicated previously, this method may also have certain disadvantages compared to the classical method. These include the time required to master the method, limitation to solution of certain types of differential equations (constant coefficients) and, in addition, a certain loss of insight into the physical problem.

In the Laplace domain, the unknown quantity to be solved for is in the general ratio of two polynomials in s. A technique, the partial fraction expansion, will now be discussed. It breaks the aforementioned ratio of polynomials into the sum of fractions. This mathematical tool is covered now, rather than later, so that the method of the Laplace transform discussed later will not be lost in mathematical details.

Consider the ratio of the following two polynomials:

$$\frac{s(s+4)}{(s+1)}(s+1)(s+2)(s+3)$$

It is desired to obtain the partial fraction expansion of the above.

$$\frac{s(s+4)}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} \qquad \text{(eq. 8)}$$

The problem is to evaluate the constants, A, B and C, of the right hand side of equation 8, an equation which is valid for all values of s.

In order to evaluate A, let s approach -1 ($s \rightarrow -1$). The term $\frac{A}{s+1}$ will become very large (approach infinity) while the other two terms, $\frac{B}{s+2}$ and $\frac{C}{s+3}$ are bounded (remain finite). Therefore, these latter two terms can be neglected compared to the term, $\frac{A}{s+1}$, on the right hand side of equation 8.

The resulting equation yields:

$$\frac{s(s+4)}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} \qquad \text{(eq. 9)}$$

$$s \rightarrow -1$$

Equation 9 results in the following since the (s + 1) term on each side of the equation may be cancelled.

$$A = \frac{s(s+4)}{(s+2)(s+3)}$$

$$s \rightarrow -1$$

$$\text{Therefore } A = \frac{-1(-1+4)}{(-1+2)(-1+3)} = \frac{-1(3)}{1(2)} = -\frac{3}{2}$$

In other words, to find the constant, A, of the fraction, A/s + 1, simply delete the factor (s + 1) from the denominator of the left-hand side of equation 8 and let s → −1. According to this rule

$$A = \underset{s \to -1}{\frac{s\,(s+4)}{\cancel{(s+1)}(s+2)(s+3)}} = \frac{-1\,(3)}{1(2)} = -\frac{3}{2}$$

In a similar fashion, to evaluate B, delete (s + 2) from the left side and let s → −2.

$$B = \underset{s \to -2}{\frac{s\,(s+4)}{(s+1)\,\cancel{(s+2)}\,(s+3)}} = \frac{-2\,(2)}{-1\,(1)} = 4$$

$$C = \underset{s \to -3}{\frac{s\,(s+4)}{(s+1)\,(s+2)}} = \frac{-3\,(1)}{-2\,(-1)} = -\frac{3}{2}$$

Therefore the partial fraction solution is:

$$\frac{s\,(s+4)}{(s+1)\,(s+2)\,(s+3)} = \frac{-\frac{3}{2}}{s+1} + \frac{4}{s+2} + \frac{-\frac{3}{2}}{s+3} \qquad \text{(eq. 10)}$$

Since equation 10 is valid for all values of s, let us pick some convenient value of s, such as zero, to check the value of the left side against the value of the right side.

$$\frac{0\,(4)}{(1)\,(2)\,(3)} = \frac{-\frac{3}{2}}{1} + \frac{4}{2} - \frac{-\frac{3}{2}}{3} \qquad (1)\,(2)\,(3)$$

$$0 = -\frac{3}{2} + 2 - \frac{1}{2}$$

$$0 = 0 \quad \underline{\text{as a check}}$$

(Do not pick a value of s equal to −1, −2 or −3 for a check, since this results in a magnitude of infinity on both sides of equation 10 and therefore will not be a check on your determined constants.)

Example 1

Obtain the partial fraction expansion for $\dfrac{6s^2 + 10s + 2}{s^3 + 3s^2 + 2s}$

$$\frac{6s^2 + 10s + 2}{s^3 + 3s^2 + 2s} = \frac{6s^2 + 10s + 2}{s\,(s^2 + 3s + 2)} = \frac{6s^2 + 10s + 2}{s\,(s + 1)\,(s + 2)}$$

$$\frac{6s^2 + 10s + 2}{s\,(s + 1)\,(s + 2)} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 2}$$

$$A = \frac{6s^2 + 10s + 2}{(s + 1)\,(s + 2)} = \frac{2}{(1)\,(2)} = 1$$

$$s \to 0$$

$$B = \frac{6s^2 + 10s + 2}{s\,(s + 2)} = \frac{6 - 10 + 2}{(-1)\,(1)} = 2$$

$$s \to -1$$

$$C = \frac{6s^2 + 10s + 2}{s\,(s + 1)} = \frac{24 - 20 + 2}{(-2)\,(-1)} = 3$$

$$s \to -2$$

$$\therefore \frac{6s^2 + 10s + 2}{s^3 + 3s^2 + 2s} = \frac{1}{s} + \frac{2}{s + 1} + \frac{3}{s + 2}$$

a check at $s = 1$ yields:

$$\frac{6 + 10 + 2}{1 + 3 + 2} = \frac{1}{1} + \frac{2}{2} + \frac{3}{3}$$

$$\frac{18}{6} = 3 \qquad\qquad 3 = 3 \text{ check}$$

Development of Laplace Transformation

The Laplace transformation from the time domain to the complex frequency, s, domain is defined as

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt \quad \text{(eq. 1)}$$

where $\mathcal{L}[f(t)]$ is, simply, the Laplace transform of the time function, f(t). Although equation 1 appears quite formidable, the technology student should not despair.

The derivation of the Laplace transform for several common inputs will now be considered.

(a) Let f(t) = K a constant or dc input

$$\mathcal{L}f(t) = F(s) = \int_0^{\infty} Ke^{-st}\, dt$$

$$= K\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{-K}{s}\left[e^{-st}\right]_0^{\infty}$$

$$= \frac{-K}{s}[\,e^{-s} \;-e^{-s(0)}]$$

therefore, $\mathcal{L}K = \frac{-K}{s}[\,0-1\,] = \frac{K}{s}$

The constant, K, in the time domain has as its corresponding Laplace transform, $\frac{K}{s}$, and the two constitute a unique transform pair.

Conversely, the inverse transformation or the inverse Laplace transformation (symbol $^{-1}$) of $\frac{K}{s}$ is K. That is, the time function whose Laplace transformation is $\frac{K}{s}$, is K.

Mathematically, $\mathcal{L}^{-1}\frac{K}{s} = K$

Note that the inverse Laplace transformation converts from the complex frequency, s, domain to the time domain.

(b) Let $f(t) = e^{-at}$

$$\mathcal{L}e^{-at} = F(s) = \int_0^{\infty} e^{-at}\, e^{-st}\, dt$$

$$= \int_0^{\infty} e^{-(a+s)t}\, dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty}$$

$$= \frac{1}{s+a}\left[-e^{-(s+a)\infty} + e^{-(s+a)0}\right]$$

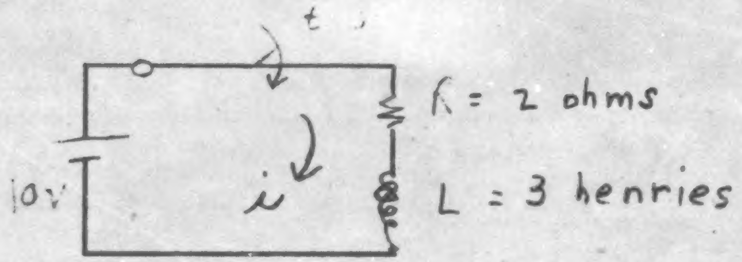
$$\mathcal{L}e^{-at} = \frac{1}{s+a}[\,0+1] = \frac{1}{s+a}$$

The technology student is made familiar with some simple derivations and then a table of useful transform pairs is presented.

Laplace Transform Pairs

Time Domain	Laplace Transform
f(t)	F(s)
1. K	$\frac{K}{s}$
2. e^{-at}	$\frac{1}{s+a}$
3. $\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
4. Kt	$\frac{K}{s^2}$
5. $\frac{d}{dt}\,[\,f(t)\,]$	$S\,F(s) - f(0_+)$
6. $\int f(t)$	$\frac{F(s)}{s} + \frac{\int f(t)\,dt}{s}$ (at $t = 0_+$)

Example 1



Find the complete solution for the current, i, using the Laplace transform method.

From electrical theory, there results

$$10 = R_i + L\frac{di}{dt} \qquad 10 = 2_i + 3\frac{di}{dt}$$

Taking the Laplace transform of both sides

$$\frac{10}{s} = 2I(s) + 3\ [\ s\,I(s) - i\,(0_+)\ ] \qquad \text{(eq. 2)}$$

(note that the variable is current, i, and Laplace transform is I(s) which reads as I of s. For simplicity the (s) – "of s" is dropped to avoid confusion with "s" times "I").

Therefore, equation 2 yields $\frac{10}{s} = 2I + 3\ [\ sI - i\,(0_+)\ ]$

i (0_+) means the value of the current at $t = 0_+$ (immediately after the switch is closed) and from electrical theory this boundary condition is zero.

$$\frac{10}{s} = 2I + 3sI = I\,(3s + 2)$$

$$I = \frac{10}{s\,(3s + 2)} = \frac{10}{3s\,(s + \frac{2}{3})}$$

$$\frac{10}{3s\,(s + \frac{2}{3})} = \frac{A}{s} + \frac{B}{s + \frac{2}{3}}$$

$$A = \frac{10}{3\,(s + \frac{2}{3})} = \frac{10}{3 \times \frac{2}{3}} = 5$$

$$s \to 0$$

$$B = \frac{10}{3s} = \frac{10}{3\,(-\frac{2}{3})} = -5$$

$$s \to -\frac{2}{3}$$

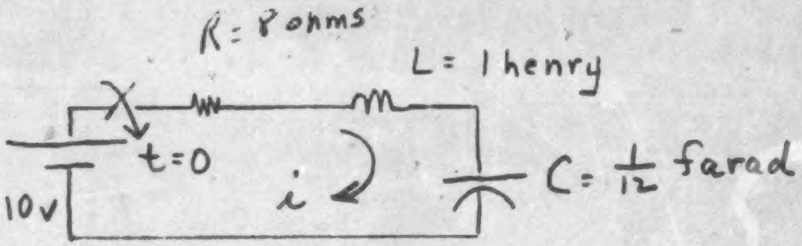
$$\therefore\ I = \frac{5}{s} - \frac{5}{s + \frac{2}{3}}$$

The inverse Laplace transform yields the current as a function of time

$$i(t) = 5 - 5e^{-\frac{2t}{3}} \qquad \text{(eq. 3)}$$

Note that equation 3 is the complete solution – where the constant term, 5, is the forced (particular) solution and the term, $-5e^{\frac{-2t}{3}}$, is the natural (complementary) solution.

Example 2



Find the complete solution for the current, i, using the Laplace transform method.

$$10 = R_i + L\frac{di}{dt} + \frac{1}{C}\int idt \qquad 10 = 8i + 1\frac{di}{dt} + 12\int idt$$

Taking the Laplace transform of both sides:

$$\frac{10}{s} = 8I + 1\,[\,sI - i\,(0_+)] + 12\,[\,\frac{I}{s} + \frac{\int idt}{s}\,(\,t = 0_+)\,]$$

from electrical theory, $i\,(0_+) = 0$

and $\int$ idt is the charge, q, which is equal to 0 at $t = 0_+$

$$\frac{10}{s} = 8I + sI + 12\,\frac{I}{s}$$

$$10 = 8sI + s^2I + 12I$$

$$10 = I\,(s^2 + 8s + 12)$$

$$I = \frac{10}{s^2 + 8s + 12} = \frac{10}{(s + 2)\,(s + 6)}$$

$$I = \frac{10}{(s + 2)\,(s + 6)} = \frac{A}{s + 2} + \frac{B}{s + 6}$$

$$A = \frac{10}{s + 6} = \frac{10}{4} = 2.5$$

$$s \rightarrow -2$$

$$B = \frac{10}{s + 2} = -\,2.5$$

$$s \rightarrow -\,6$$

$$I = \frac{2.5}{s + 2} - \frac{2.5}{s + 6}$$

$$\therefore\ \pounds\, I^{-1}(s) = \underline{i\,(t)} = \underline{2.5e^{-2t} - 2.5e^{-6t}}\ \text{(solution)}$$

The forced response is zero (which checks electrical theory) and the complete solution consists of the natural response.

An area of importance in the study of the stability of automatically controlled (feedback) systems is the analysis of Bode plots.

Bode graphs are straight line plots of a number, N (gain) in decibels ($20 \log_{10} N$) versus the log of the frequency (or angular velocity). Of particular interest is the response of the system to a sinusoidal input ($s = j\omega$). The Bode plots give a solution in both magnitude and phase angle. Semi-log paper of the appropriate number of cycles is necessary.

(a) Consider the transfer function, $G(s) = \frac{K}{s}$ for $s = j\omega$

$$G(j\omega) = \frac{K}{j\omega}$$

$$|G| = \frac{K}{\omega} \text{ (since the magnitude of } j\,\omega = (\omega))$$

$$|G|db = 20 \log \frac{K}{\omega} = 20 \log K - 20 \log \omega$$

$$= -20 \log \omega + 20 \log K \quad \text{(eq. 4)}$$

It is to be remembered that the ordinate will be the gain, G, in decibels versus the abscissa, the log of the frequency, ω.

Equation 4 is the plot of a straight line ($y = m\,x + b$) with a negative slope (when plotted on semi-log paper).

To determine the slope in the proper units, let $\omega = 1$ and then 10; this change by a factor of 10 is called a decade.

A change by a factor of 2 is called an octave.

$$\omega = 1 \; |G|_{db} = 20 \log K - 20 \log 1 = 20 \log K$$

$$\omega = 10 \; |G|_{db} = 20 \log K - 20 \log 10 = 20 \log K - 20$$

$$\therefore \text{Slope} = \frac{\Delta \text{Vert}}{\Delta \text{Horiz}} = \frac{20\text{db}}{\text{decade}}$$

$$\omega = 2 \; |G|_{db} = 20 \log K - 20 \log 2 = 20 \log K - 20\,(.30)$$

$$= 20 \log K - 6$$

$$\therefore \text{Slope} = -\frac{6\text{db}}{\text{octave}}$$

In other words, a slope of −6db/octave is equivalent to a slope of −20db/decade.

To plot the straight line, the abscissa (x) intercept is obtained. At this point the ordinate (G in decibels) would be zero.

$$\therefore |G|_{db} = 0 = 20 \log \frac{K}{\omega} = 20 \log 1$$

The ordinate in decibels will be zero when the number is unity or when $\omega = K$ which is the ω intercept.

To obtain the phase angle plot —

$$G(j\,\omega) = \frac{K}{j\omega} = \frac{K}{\omega\angle 90} = \frac{K}{\omega}\angle -90$$

which indicates that for all frequencies the phase angle is −90 degrees (this indicates the output will lag the input by 90 degrees).

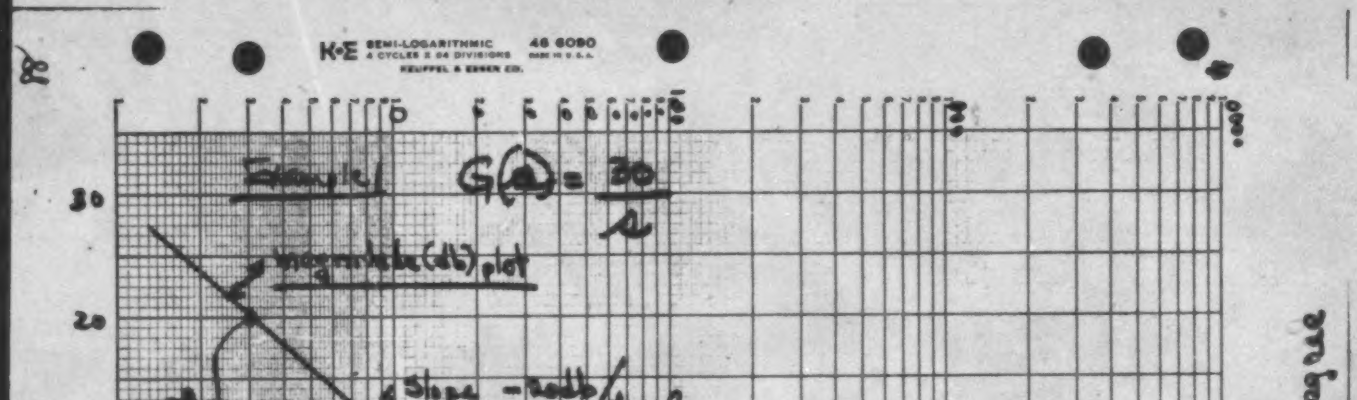
Example 1

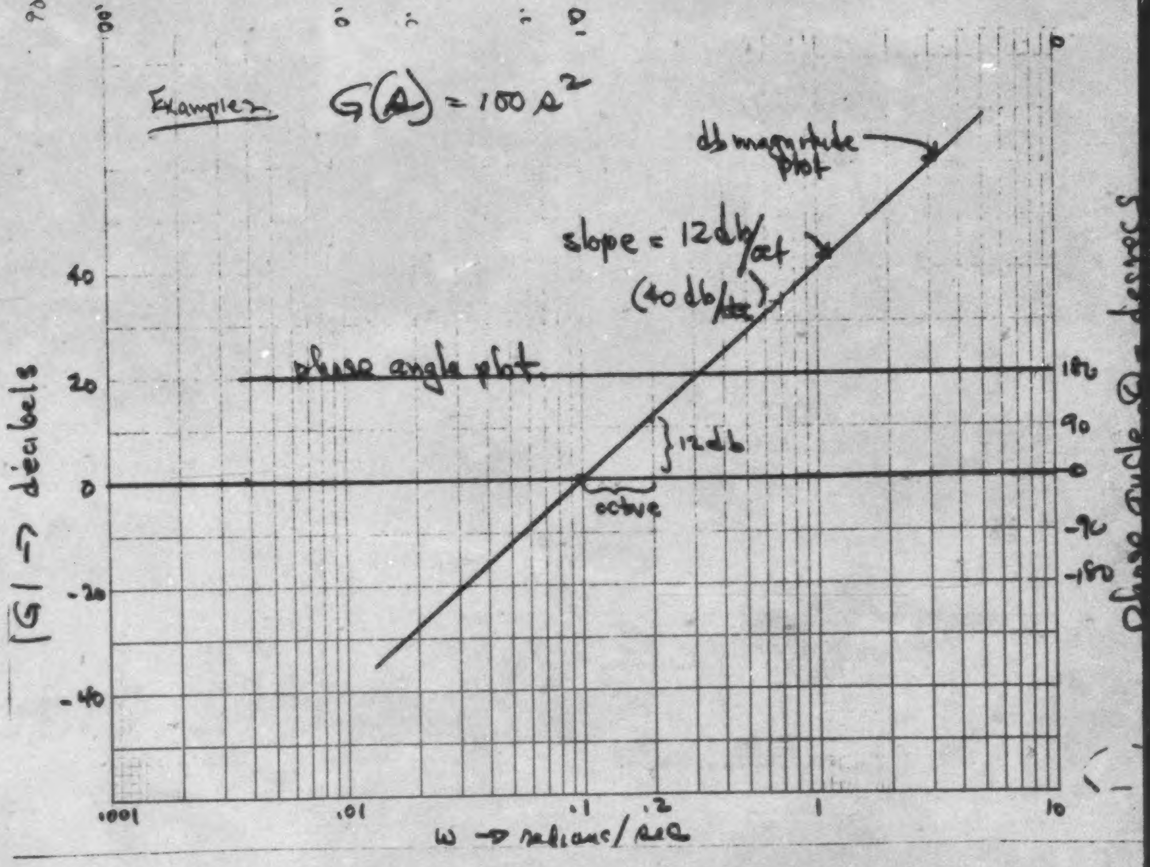
Given G(s) = 30/s, plot the magnitude and phase angle for all frequencies (Bode plot).

The "ω" intercept is 30 (at 0 db for the ordinate, G), and the slope is $\frac{-20\text{db}}{\text{decade}}$ or $\frac{-6\text{ db}}{\text{octave}}$

The phase angle is −90 degrees for all frequencies (see below).

For a transfer function of G(s) = Ks, it is easily seen that the straight line (Bode) plot would have a positive slope of $\frac{20\text{ db}}{\text{decade}}$ or $\frac{6\text{ db}}{\text{octave}}$ and the "ω" intercept (0 db for G) would be at $\frac{1}{K}$. The phase angle would be constant at plus 90 degrees.




Example 2. G(s) = 100 s^2
db magnitude plot
slope = 12db/oct
(40 db/dec)
phase angle plot
12db
octave
|G| → decibels
40
20
0
-20
-40
phase angle θ → degrees
180
90
0
-90
-180
.001
.01
.1
.2
1
10
ω → radians/sec

Example 2

$G(s) = 100s^2$

Obtain Bode-magnitude and phase angle - plots.

The slope is positive (“s)) in the numerator of the transfer function) and the s^2 will yield twice the slope for “s” or $\frac{40\text{ db}}{\text{decade}}$ or $\frac{12\text{ db}}{\text{octave}}$

The “ω” intercept will occur when the db = 0 or the magnitude of G is unity.

For the intercept therefore

$$100\,\omega^2 = 1 \qquad \omega^2 = \frac{1}{100} \qquad \omega = \frac{1}{10} = 0.1$$

The phase angle is constant at 180 degrees. The plot is shown on the semi-log paper.

(b) Consider the transfer function

$$G(s) = K(s + a) \quad \text{for } s = j\omega \quad \text{then} \quad G(j\omega) = K(a + j\omega)$$

At low frequencies ($\omega \ll a$) then $G(j\omega) = Ka$, a constant (horizontal line) and the magnitude of $G(j\omega)$ in decibels in equal to 20 log Ka. The phase angle of $G(j\omega)$ for these frequencies is zero since $G(j\omega)$ is a real positive number.

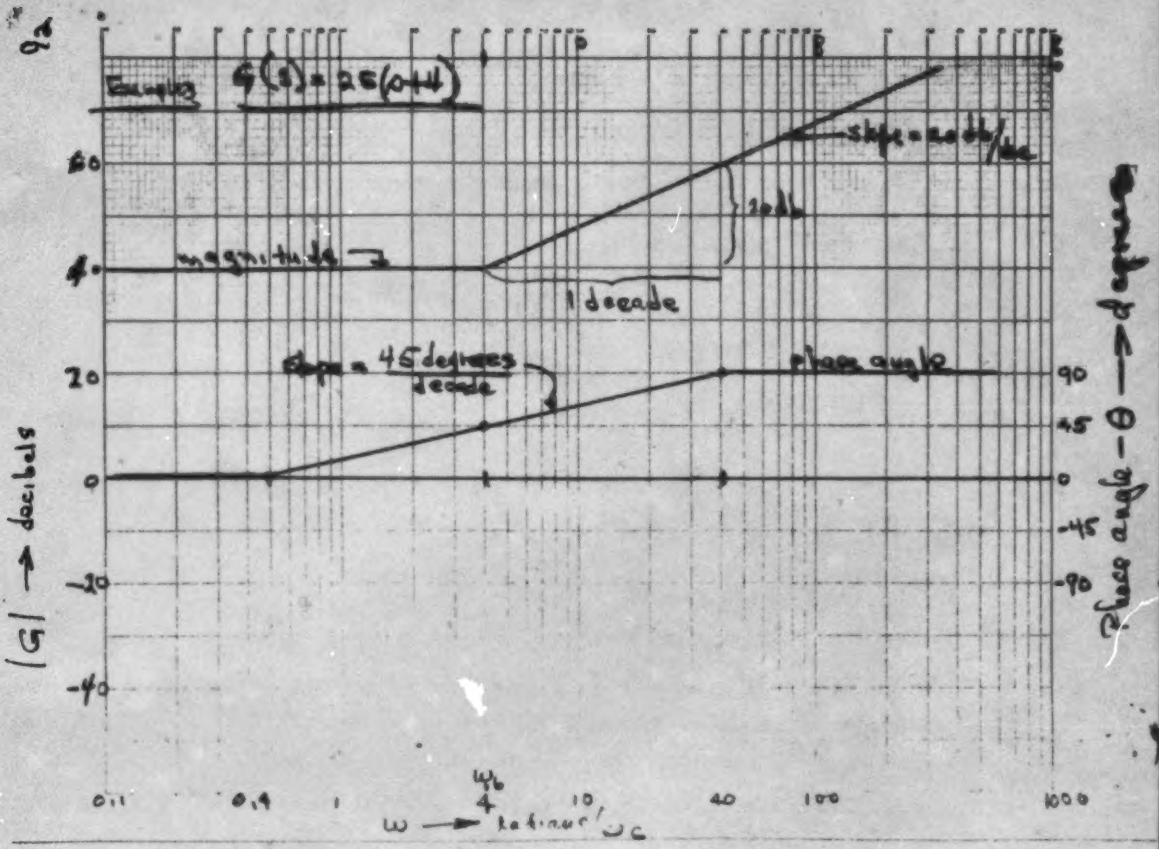
At high frequencies ($\omega \gg a$) then $G(j\omega) = K j\omega$ or $G(s) = Ks$ which is a straight line of positive slope of 20 db/decade or 6 db/decade. The phase angle is positive 90 degrees, since $G(j\omega)$ is a purely imaginary number ($jK\omega$ with no real component).

At some frequency of ω (between a “low” and “high” value) the line changes from a horizontal line ($\frac{0\text{ db}}{\text{decade}}$) to that of a straight line having a positive slope of $\frac{20\text{ db}}{\text{decade}}$.

The frequency at which this change in slope occurs is called the break frequency. This frequency can be obtained mathematically as the intersection of the two straight lines $|G(j\omega)| = Ka$ and $|G(j\omega)| = |jK\omega| = K\omega$.

$$Ka = K\omega \quad \text{or} \quad \omega_b = a \text{ (break frequency)}$$

The phase angle of $G(j\omega)$ at this break frequency is 45 degrees since $G(j\omega) = K(j\omega + a) = K(a + ja)$ and the real and imaginary components of the complex number are equal.



Example 3

Obtain the magnitude (db) and phase angle plots of the transfer function,

$G(s) = 25(s+4)$

$$G(j\omega) = 25(4 + j\omega)$$

At low frequencies $|G(j\omega)| = 100 \;(\omega \ll 4)$

and in db, $G(j\omega) = 20 \log 100 = 40$ db

The break frequency occurs at $\omega = 4 \frac{\text{radians}}{\text{sec.}}$

when the line changes from a horizontal line to one with a positive slope of $20 \frac{\text{db}}{\text{octave}}$. The plot is shown on semi-log paper. The phase angle at $\omega = 4$ is 45 degrees and is zero at "low" frequencies and 90 degrees at "high" frequencies. For practical engineering purposes the angle is zero degrees a decade below $\omega = 4$ and 90 degrees a decade above $\omega = 4$.

From the basic fundamentals of Bode plots, more complex transfer functions can be dealt with, quickly and easily.

In conclusion, mathematics teachers in technology should eschew the "abstract concepts syndrome" and apply the mathematical principles to practical problems. Of paramount importance is the principle that these instructors should have some familiarity with technology terms.

For instance, the terms, "poles" and "zeros" should not conjure up visions of clotheslines and grades on mathematics exams, but should bring to mind, respectively, values of "s" which make the denominator of a transfer function zero and the values of "s" which make the numerator of a transfer function zero.

ENGINEERING TECHNOLOGIES—MECHANICAL AND DESIGN DRAFTING

Technicians play essentially a supporting role, often requiring close work with engineers, scientists and other professional personnel. They are doers in the many roles of the engineering and scientific teams. Engineers reduce scientific laws and principles to working applications—technicians apply these applications to actual situations. Scientists explore ideas—technicians perform specific details. Technicians, therefore, play an important part in converting ideas into accomplishments, whether in a laboratory or drafting room.

Mechanical Technology (M.T.)

As part of the specialized engineering group the mechanical technician takes apart new ideas and tests them to see how they can be applied to developing new products and production methods.

Industrial production technicians assist in developing new and improved production methods and procedures in manufacturing plants. Their duties include designing automated systems, conducting time-and-motion studies, planning work flow, quality control, and additional specialties such as industrial safety.

Other mechanical technicians who work in manufacturing include air conditioning and refrigeration technicians and automotive technicians. Air conditioning technology involves heating, humidity, cleanliness, and movement, as well as cooling.

Mechanical technicians may work with engineers on the design, development and production of aircraft, helicopters, rockets, and spacecraft. For example, a technician under the direction of an engineer might estimate weight and other factors affecting the load capacity of an airborne body.

Technicians often work with engineers in the field of power generation and transmission. This may involve assisting in massive power plants that serve the needs of large urban areas, or it may require designing new devices such as portable power systems for use in remote areas.

Some mechanical technicians help engineers with the design and development of machinery and other equipment; specialists in this category are diesel technicians, machine designers and tool designers.

Mechanical technicians aid in research and development, design, testing and production. They also can function in the installation, operation and maintenance of mechanical equipment, supervision of the assembly of prototypes, selling of mechanical components and machines, and technical writing.

The Mechanical Technology curriculum at Queensborough Community College is broadly based in mechanical studies and is laboratory-oriented. Sufficient theory is given in the lecture sessions to enable the student to understand the basics of design. Mathematics and science courses give a sound background to the technician in the sources of mechanical technology. Practical and laboratory work are emphasized throughout the program. The mechanical graduate is thus prepared to qualify for entry-level technician jobs immediately after graduation.

Many M.T. graduates continue their higher education in engineering schools towards a

Bachelor of Engineering Technology. These four-year curriculums introduce more theory and additional mathematics.

The mechanical courses are designed to provide the technician with the analytical tools needed to perform the work in the many areas of the mechanical field. The M.T. curriculum consists of three major lines of study: Manufacturing Processes and Systems, Thermo-Fluid Mechanics and Machine Design. Supporting courses in Technical Drawing, Metallurgy and Materials, Applied Mechanics, Strength of Materials, Computer Applications and Principles of Electrical Technology fill out the curriculum in the technical vein. One-and-a-half years of technical mathematics and 1 year of physics supplement the technical studies.

Design Drafting (D.D.)

Closely related to engineering technicians, and considered in the technician category, are the draftsmen. Draftsmen translate the ideas, rough sketches, specifications and calculations of engineers into working plans for making a product or structure. Included in the many categories of drafting are product design drafting, styling drafting, architectural design drafting, technical illustrating, and patent drawing. Draftsmen are employed by industrial organizations, government and business.

Drafting is a language or picture-writing—a most important form of communication in the technical field. It is the skill which transfers designs into lines and dimensions on paper. Most drafting procedures require translating every detail of the three-dimensional object into two-dimensional drawings and some drafting is symbolic, as in wiring and piping drawing.

As an example of the great need for draftsmen, more than 27,000 drawings are required to build an average passenger car which is made up of about 12,000 parts. These include body drawings which picture sheet metal surfaces, mechanical drawings which show the size and shape of each part and their assembly into an automobile, as well as drawings which show the tools necessary to make and assemble the various parts.

The D.D. curriculum at Queensborough Community College uses a less analytic approach than the M.T. curriculum. The program as such is oriented towards visualization and the expression of objects on paper with some basics in introductory design. However, many D.D. graduates may be called upon to do some work which will require an understanding and manipulation of mathematics. With experience, D.D. graduates can achieve the role of a senior designer which will require further applications of mathematical skills as the design problems increase in complexity.

There are opportunities for the Design Drafting graduates to continue their higher education at four-year schools and several graduates have gone on to schools of engineering and architecture.

The D.D. curriculum is made up of 1 year of basic graphics (Technical Descriptive Geometry and Drawing), 1 year of analytical courses (Elements of Technology and Statics and Strength of Materials) and advanced drawing courses spanning all the various areas that require draftsmen to have a knowledge of the specialized designations and specifications as used in those fields. Courses in Selection of Materials and Manufacturing Processes or Surveying round out the technical offering in the curriculum. One year of design mathematics and ½ year of physics supplement the design drafting courses.

It should be noted that the D.D. curriculum allows the student to choose 3 courses relating to the manufacturing area (Manufacturing Processes, Tool and Die Design, Mechanisms) or to the architectural-construction area (Surveying, Construction Methods, Architectural Design).

Mathematics For Mechanical Technology and Design Drafting Technology

Several courses in the M.T. and D.D. curricula will now be briefly described as to their content and objectives, each course to be followed by sample problems. These selected sample problems are mainly to display the use and types of mathematics required in their solution.

In summary, the courses are arranged as follows:

Mechanical Technology

(1) Applied Mechanics
(2) Strength of Materials
(3) Fluid Mechanics
(4) Thermodynamics
(5) Machine Design
(6) Manufacturing Systems
(7) Introduction to Numerical Control

Design Drafting

(1) Elements of Technology
(2) Statics and Strength of Materials
(3) Surveying and Layout
(4) Technical Drawing
(5) Piping Systems
(6) Duct Systems
(7) Mechanisms
(8) Architectural Fundamentals
(9) Construction Methods
(10) Architectural Design
(11) Structural Drafting and Design

Applied Mechanics

Mechanics is the science of determining the forces in mechanical devices and structures and the effect of the forces upon the bodies that make up these devices and structures.

The study of mechanics leads to an understanding of the force relationships and distribution when objects are at rest or moving at a constant velocity (statics) and the changing velocities of objects caused by forces acting upon them (dynamics).

The analytical nature of the course requires a rigorous understanding of algebra and trigonometry. In particular, quadratic equations, simultaneous equations, the law of sines, similar triangles and proportions are among the mathematical topics that are necessary to solve many of the problems that occur in mechanics.

The following illustrative problems will show the application of mathematical skills to the various types of problems found in mechanics.

A force of 200 lb is directed as shown in Fig. 2-3. Determine the X and Y components of the force.

Solution: By projecting the force upon the axes, we discover that the sign of F_x is minus and of F_y positive. Applying Eq. (2-1), we obtain

$[F_x = F\cos\theta_x]$ $\quad F_x = -200\cos 30^\circ = -200 \times 0.866$

$F_x = -173.2\text{ lb}$ *Ans.*

$[F_y = F\sin\theta_x]$ $\quad F_y = 200\sin 30^\circ = 200 \times 0.5$

$F_y = +100\text{ lb}$ *Ans.*

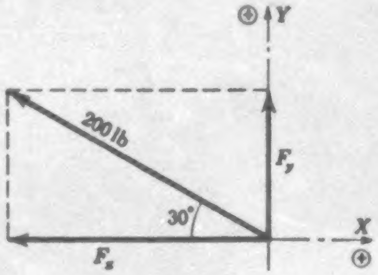
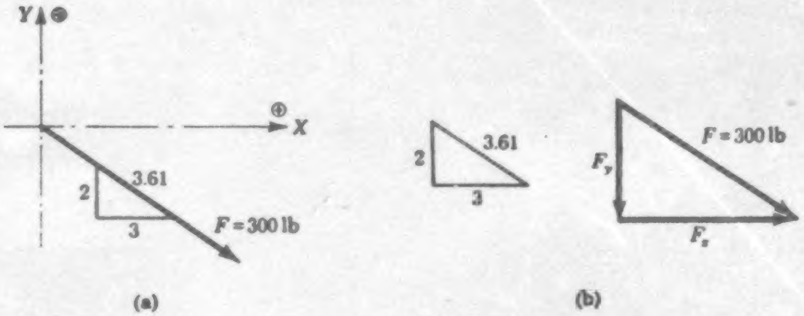


FIG. 2-3.

. Determine the components of the 300-lb force directed down to the right at a slope of 2 to 3 as shown in Fig. 2-4a.

Solution: The major difference between this problem and the preceding one is that the direction of the force is defined by its slope instead of its angle. We can compute θ_x from its tangent and then substitute its sine and cosine functions into



(a) (b)

FIG. 2-4.

Eq. (2-1), but it is simpler and more direct to compute the hypotenuse of the slope triangle as $\sqrt{(2)^2 + (3)^2} = \sqrt{13} = 3.61$ and then apply the definitions of sine and cosine as follows:

$[F_x = F\cos\theta_x]$ $\quad F_x = 300 \times \dfrac{3}{3.61} = 249\text{ lb}$

$[F_y = F\sin\theta_x]$ $\quad F_y = -300 \times \dfrac{2}{3.61} = -166\text{ lb}$

An even better procedure is to note the similarity between the slope triangle and the force triangle in Fig. 2-4b whose corresponding sides are proportional to each other. This gives

$$\frac{F_x}{3} = \frac{F_y}{2} = \frac{300}{3.61}$$

whence

$$F_x = 249\text{ lb} \quad\text{and}\quad F_y = -166\text{ lb}$$

The components of a certain force are defined by $F_x = 300$ lb and $F_y = -200$ lb. Determine the magnitude, inclination with the X axis, and pointing of the force.

Solution: The magnitude of the force is found by applying the first of Eq. (2-2).

$$\left[F = \sqrt{(F_x)^2 + (F_y)^2}\right] \qquad F = \sqrt{(300)^2 + (200)^2} \qquad F = 361 \text{ lb} \quad \textit{Ans.}$$

The inclination with the X axis is determined by the second part of Eq. (2-2).

$$\left[\tan \theta_x = \frac{F_y}{F_x}\right] \qquad \tan \theta_x = \frac{200}{300} = 0.667 \qquad \theta_x = 33.7° \quad \textit{Ans.}$$

Note particularly that by neglecting the given signs of the components the angle found is the acute angle between the force and the X axis. The direction of the force is found by sketching a tip-to-tail summation of the components as shown in Fig. 2-5, or by visualizing it mentally. Note that the minus sign of F_y indicates it to be directed downward. Hence the force F points down to the right.

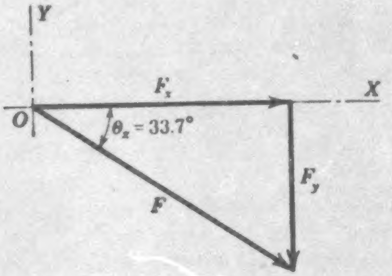


Fig. 2-5.

This technique of determining a force eliminates the necessity of remembering certain arbitrary conventions. For example, a mathematical convention defines an angle as always measured in a counterclockwise sense from the X axis. Accordingly in the given example, θ_x might be defined as $-33.7°$ or as $+326.3°$.

Resultant of Parallel Forces — Illustrative Problems

Determine the resultant of the parallel force system acting on the bar AB shown in Fig. 2-19. The forces and positions are given in the figure.

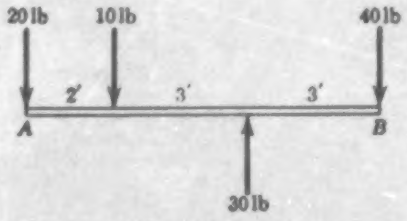


FIG. 2-19.

Solution: The magnitude of the resultant force is first obtained by applying

$$[R = \Sigma F] \qquad R = -20 - 10 + 30 - 40$$
$$R = -40 \text{ lb} \quad \textit{Ans.}$$

Upward forces having been assumed to be positive, the negative sign of R indicates it to be directed downward.

Applying the principle that the moment of the resultant is equal to the moment sum of its parts (Varignon's theorem, Art. 2-5), we have, taking clockwise moments about A as positive.

$$[M_R = \Sigma M_A] \qquad \circlearrowright \Sigma M_A = 10 \times 2 - 30 \times 5 + 40 \times 5$$
$$= 190 \text{ lb-ft} \circlearrowright$$
$$[M_R = R \cdot d = \Sigma M_A] \qquad 40\, d_A = 190 \qquad d_A = 4.75 \text{ ft} \quad \textit{Ans.}$$

To determine the position of R relative to A, draw R acting downward (because of the minus sign) as shown in Fig. 2-20. Since the moment sum of the original forces was found to be clockwise, R must lie to the right of the moment center A in order also to produce a clockwise moment.

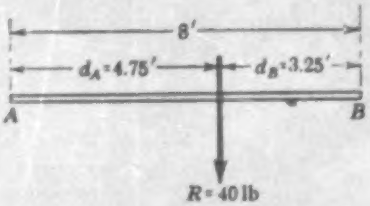


FIG. 2-20.

By locating R with respect to B, it is easily shown that the position of the resultant is independent of the choice of moment center. Thus we have

$$[M_R = \Sigma M_B] \qquad \circlearrowright \Sigma M_B = -20 \times 8 - 10 \times 6 + 30 \times 3$$
$$= -130 \text{ lb-ft} \circlearrowleft$$
$$[M_R = R \cdot d = \Sigma M_B] \qquad 40\, d_B = 130 \qquad d_B = 3.25 \text{ ft} \quad \textit{Ans.}$$

Referring to Fig. 2-20 and noting that the sign of ΣM_B is negative (thereby indicating a counterclockwise moment), we see that R must lie to the left of the moment center B to create an equivalent counterclockwise moment. Moreover, $d_A + d_B = 4.75 + 3.25 = 8$ ft, which is the total distance from A to B. Hence the position of R is independent of the choice of moment center.

It is usually convenient to choose the moment center somewhere near the middle of the given system of forces in order to simplify calculations by having smaller moment arms. Also, it is wise to select the moment center at one of the forces in order to eliminate the moment effect of that force from the computations.

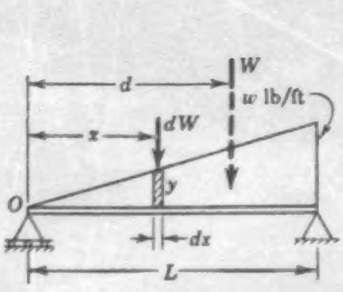


FIG. 2-21.

A beam of length L supports a load which varies from w lb per ft at the right end to zero at the left end. Determine the magnitude and position of the resultant load.

Solution: The total weight W of the triangular load shown in Fig. 2-21 is the resultant of smaller parallel loads like dW each of which is the product of an intensity of y lb per ft by the small length of dx ft along which it may be assumed constant. From the proportionality between similar triangles, we have $\frac{y}{x} = \frac{w}{L}$ or $y = \frac{w}{L}x$. Applying Eq. (2-5) we obtain

$$[R = \Sigma F] \qquad W = \int_0^L y\,dx = \frac{w}{L}\int_0^L x\,dx = \frac{wL}{2} \quad \textit{Ans.}$$

The position of this resultant weight from O is obtained from Eq. (2-6):

$$[R \cdot d = \Sigma M_O] \qquad \frac{wL}{2}\,d = \int_0^L x(y\,dx) = \frac{w}{L}\int_0^L x^2\,dx = \frac{wL^2}{3}$$

whence

$$d = \tfrac{2}{3}L \quad \textit{Ans.}$$

Equilibrium of Concurrent Force Systems — Illustrative Problems

A system of cords knotted together at A and B support the weights shown in Fig. 3-4. Compute the tensions P, Q, F, and T acting in the various cords.

Solution: We begin by drawing a FBD of knots A and B. Of these two concurrent force systems, we must first solve that at A. The force system at B is temporarily indeterminate because it contains three unknown forces and has available only two independent equations of equilibrium. Its solution must be postponed until one of the unknowns, P in this instance, has been determined from the concurrent system acting at A, where P, exerting an equal and opposite effect to its action on B, is only one of two unknowns.

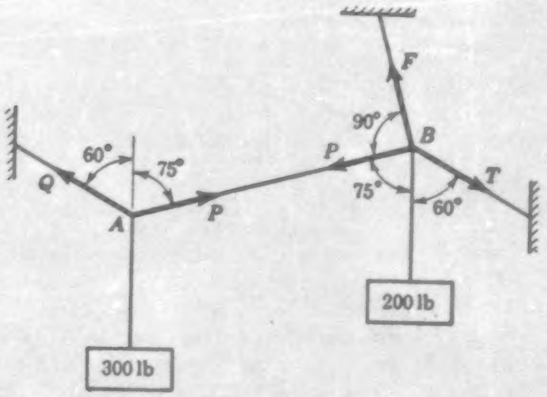


FIG. 3-4.

Several methods are available for the solution of the concurrent force system at A. Let us discuss each of these methods so that their individual advantages or disadvantages will enable us to select the most efficient and rapid method to use in similar problems.

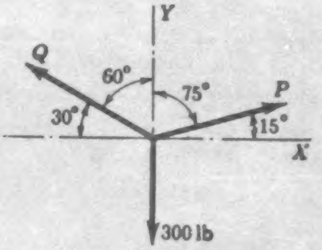


Fig. 3-5. — Method I.

Method I—Using Horizontal and Vertical Axes. This is a routine method requiring no imagination. Selecting reference axes that are horizontal and vertical as shown in Fig. 3-5, we apply the conditions of equilibrium, Eq. (3-1), to obtain

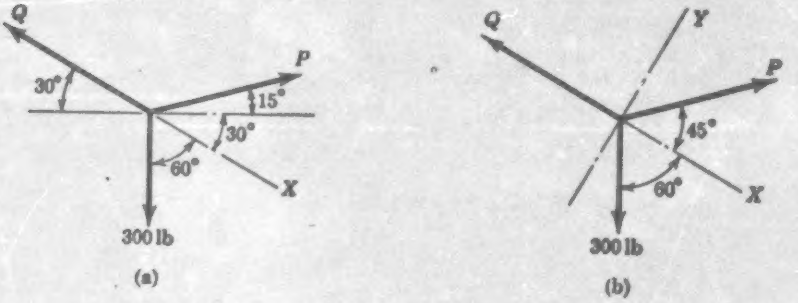
$$[\Sigma X = 0] \qquad P\cos 15^\circ - Q\cos 30^\circ = 0 \qquad (a)$$

$$[\Sigma Y = 0] \qquad P\sin 15^\circ + Q\sin 30^\circ - 300 = 0 \qquad (b)$$

Solving Eqs. (a) and (b) simultaneously yields

$$\left.\begin{aligned} P &= 367 \text{ lb} \\ Q &= 410 \text{ lb} \end{aligned}\right\} \quad \textit{Ans.}$$

Method II — Using Rotated Axes. The disadvantage of Method I is the necessity of solving simultaneous equations. Since the reference axes are arbitrarily selected in the first place, a better choice of the reference axes will eliminate simultaneous equations; this simplifies the numerical work and reduces the chance for error. For example, let the X axis be selected to pass through one of the unknowns, say Q. In this case Q will have no Y component and will not appear in a Y summation.



(a)

(b)

Fig. 3-6. — Method II: Using rotated axes.

The method of determining the angles between the forces and the rotated reference axes is shown in Fig. 3-6a; the final values of the angles are shown in Fig. 3-6b. When actually solving the problems, only the X axis need be drawn, as in Fig. 3-6a. The Y axis can be omitted; it is understood to be perpendicular to the X axis.

Since the X axis was chosen to coincide with Q, it is evident that Q has no Y component. Hence by applying the condition of equilibrium, $\Sigma Y = 0$, we automatically eliminate Q from the equation. Thus we have

$$[\Sigma Y = 0] \qquad P\sin 45^\circ - 300\sin 60^\circ = 0 \qquad P = 367 \text{ lb} \quad \textit{Ans.}$$

Having determined P, we readily find the second unknown Q by applying the second equation of equilibrium:

$$[\Sigma X = 0] \qquad 367\cos 45^\circ + 300\cos 60^\circ - Q = 0 \qquad Q = 410 \text{ lb} \quad \textit{Ans.}$$

Note carefully the technique used. When the X axis is chosen so that it coincides with one of the unknowns, the Y summation determines the other unknown. Then the X summation determines the remaining unknown.

Method III — Using Force Triangle. When three forces are in equilibrium, the easiest solution is generally obtained by applying the sine law to the triangle representing the polygon of forces. Since forces in equilibrium have a zero resultant, the tip of the last vector must touch the tail of the first vector. This tip-to-tail addition gives the closed polygon of forces shown in Fig. 3-7. Applying the law of sines to this triangle, we obtain[1]

$$\frac{300}{\sin 45^\circ} = \frac{P}{\sin 60^\circ} = \frac{Q}{\sin 75^\circ}$$

whence as before

$$P = 367 \text{ lb} \quad \text{and} \quad Q = 410 \text{ lb} \quad \textit{Ans.}$$

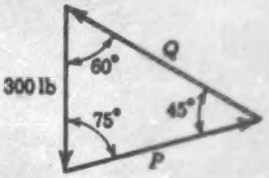


FIG. 3-7. — Method III: Using force triangle.

We are now ready to determine the forces F and T holding the concurrent system of forces at B in equilibrium. A closed polygon of forces for this system forms a quadrilateral so that the sine law cannot be applied. Although a diagonal of this quadrilateral can be drawn that will subdivide it into two triangles to which the sine law can be applied, this procedure is more cumbersome than the method of using rotating axes described above in Method II.

Applying the method of rotated axes to the FBD of B, we draw the X axis to coincide with T as in Fig. 3-8, thereby eliminating T from a Y summation. Hence we obtain F from

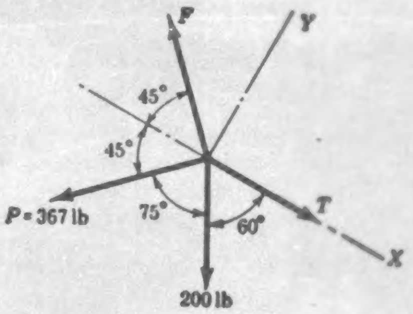


FIG. 3-8. — Method of rotated axes applied to FBD of B.

$$[\Sigma Y = 0] \qquad F \sin 45^\circ - 367 \sin 45^\circ - 200 \sin 60^\circ = 0 \qquad F = 612 \text{ lb} \quad \textit{Ans.}$$

The remaining unknown T is now determined from

$$[\Sigma X = 0] \qquad T + 200 \cos 60^\circ - 367 \cos 45^\circ - 612 \cos 45^\circ = 0$$

$$T = 593 \text{ lb} \quad \textit{Ans.}$$

Concurrent Force System – Illustrative Problem

The bell crank shown in Fig. 3-12a is supported by a bearing at A. A 100-lb force is applied vertically at C, rotation being prevented by the force P acting at B. Compute the value of P and the bearing reaction at A.

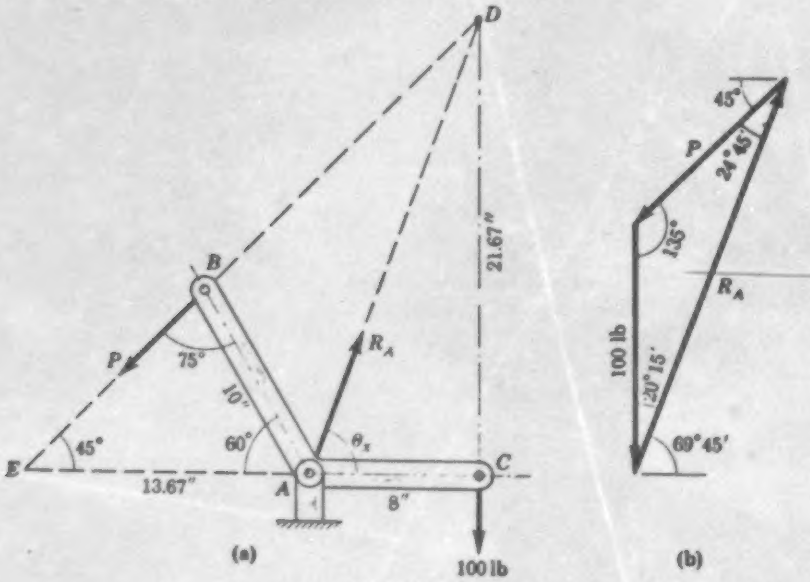


FIG. 3-12.

Solution: Since the bell crank is in equilibrium, the three forces which act upon it must pass through a common point. Prolonging the lines of action of the forces to intersect at D makes the direction of R_A such that it must pass through A and D. From the geometry of the figure, the distance AE is found to be 13.67 in., whence the distance CD = 21.67 in. The direction of R_A is found from

$$\left[\tan\theta_x = \frac{DC}{AC}\right] \qquad \tan\theta_x = \frac{21.67}{8} = 2.71 \qquad \theta_x = 69°\,45' \quad \textit{Ans.}$$

Plotting the polygon of forces that are acting on the bell crank as shown in Fig. 3-12b, we obtain by applying the law of sines

$$\frac{100}{\sin 24°\,45'} = \frac{P}{\sin 20°\,15'} = \frac{R_A}{\sin 135°}$$

whence $P = 82.8$ lb and $R_A = 169$ lb *Ans.*

If desired, the value of P may be checked by taking moments about A. Then we have

$$[\Sigma M_A = 0] \qquad (P \sin 75°) \times 10 - 100 \times 8 = 0 \qquad P = 82.8 \text{ lb} \quad \textit{Check}$$

The moment of P about A was obtained by applying Varignon's principle (see Fig. 2-15). By resolving the force P into components parallel and perpendicular to AB, the parallel component is made to pass through the moment center, whence the moment effect of P is due only to the perpendicular component.

Trusses – Method of Joints – Illustrative Problem

A Fink truss is loaded as shown in Fig. 4-6. Determine the force in each member of the truss assuming them to be pin-connected.

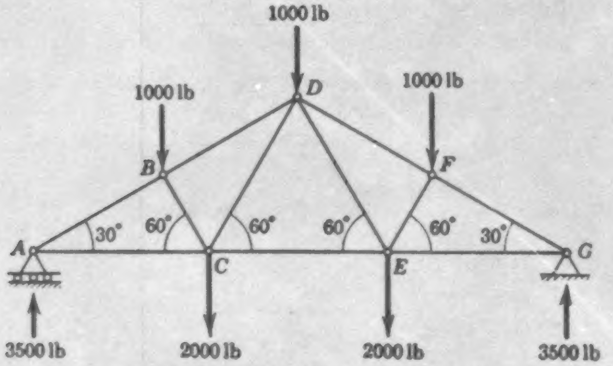
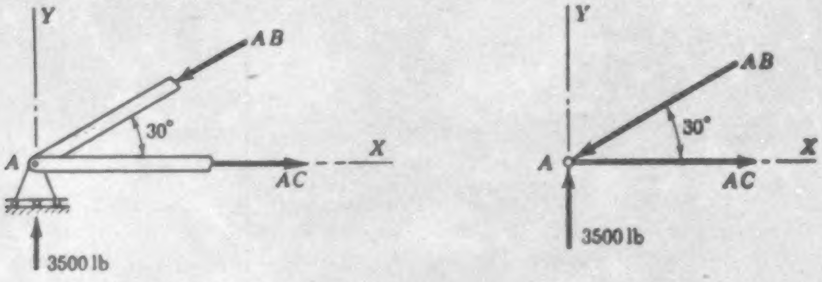

1000 lb
1000 lb
D
1000 lb
B
F
A
30°
60°
60°
60°
60°
30°
G
C
E
3500 lb
2000 lb
2000 lb
3500 lb

Fig. 4-6. — Fink truss.

Solution: The given truss is symmetrical and also symmetrically loaded so that the forces need be found in only one-half of it.

After determining the reactions from symmetry, consider joint A which has only two unmarked members (AB and AC) acting upon it. As shown in Fig. 4-7, we may use either the FBD of the joint or the equivalent FBD of the pin. Of the two, the FBD of the pin is preferred since it is simpler to draw. In either diagram, it is evident that AB denotes compression, i.e., is directed toward the pin, in order that its vertical component may balance the upward reaction. Hence AC must be in tension and pull away from the pin to balance the leftward component of AB. Selecting the X axis to coincide with the unknown force AC, we obtain


Y
AB
30°
A
X
AC
3500 lb
Y
AB
30°
A
X
AC
3500 lb

(a) FBD of joint A

(b) FBD of pin A

Fig. 4-7. — Free-body diagrams of joint and pin.

$[\Sigma Y = 0]$ $\quad 3500 - AB \sin 30° = 0 \quad$ $AB = 7000$ lb C *Ans.*

$[\Sigma X = 0]$ $\quad AC - 7000 \cos 30° = 0 \quad$ $AC = 6062$ lb T *Ans.*

The positive values obtained for AB and AC confirm the original assumption concerning the direction of these forces. The action of members AB and AC on their end pins, indicating respectively compression and tension, may now be drawn as shown in Fig. 4-8. (In an actual problem the arrows would be placed on the

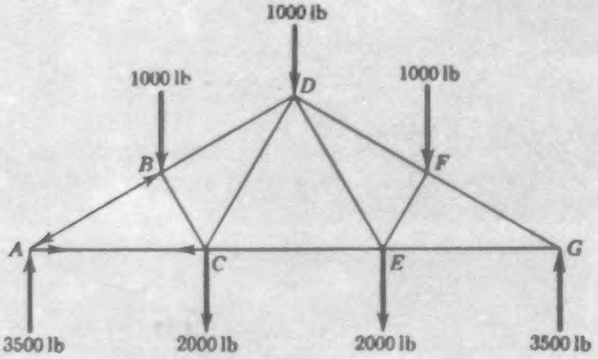

1000 lb
1000 lb
D
1000 lb
B
F
A
C
E
G
3500 lb
2000 lb
2000 lb
3500 lb

Fig. 4-8. — Truss marked to show effect of members AB and AC on their end pins.

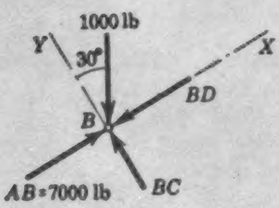


FIG. 4-9. — Free-body diagram of pin *B*.

original diagram of the truss, but to indicate the marked and unmarked members more clearly, the truss is here redrawn.)

From Fig. 4-8, the next pin at which no more than two unmarked members appear is seen to be *B*. Repeating the technique used at pin *A*, equilibrium of pin *B* can be achieved by assuming *BD* and *BC* to be in compression and therefore acting toward the pin as shown in Fig. 4-9. Rotating the *X* axis to coincide with the unknown force *BD*, we obtain

$[\Sigma Y = 0]$ $\quad BC - 1000 \cos 30° = 0 \quad BC = 866$ lb C *Ans.*

$[\Sigma X = 0]$ $\quad 7000 - 1000 \sin 30° - BD = 0 \quad BD = 6500$ lb C *Ans.*

The positive values obtained for *BD* and *BC* confirm the fact that these forces are compressions. The action of *BD* and *BC* upon their end pins may now be marked on the original truss diagram as in Fig. 4-10.

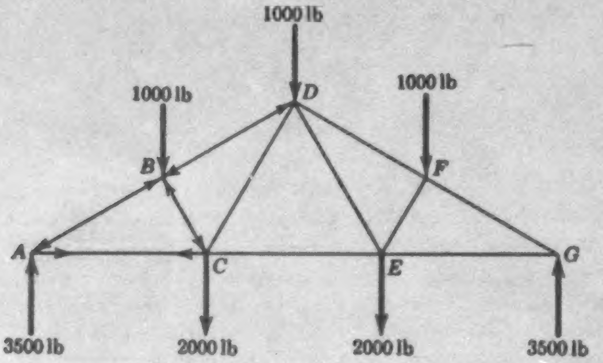


FIG. 4-10. — Truss marked to show effect of members *BD* and *BC* on their end pins.

The next pin at which two unmarked members appear is *C*. Assume both *CD* and *CE* to be in tension. The FBD of pin *C* can now be drawn as in Fig. 4-11. Selecting the *X* axis to coincide with *CE*, we have

$[\Sigma Y = 0]$ $\quad CD \sin 60° - 866 \sin 60° - 2000 = 0 \quad CD = 3175$ lb T *Ans.*

$[\Sigma X = 0]$ $\quad CE + 3175 \cos 60° + 866 \cos 60° - 6062 = 0 \quad CE = 4040$ lb T *Ans.*

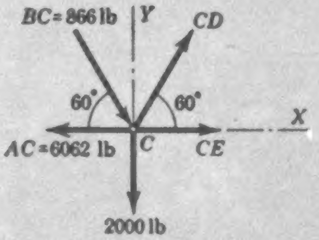


FIG. 4-11. — Free-body diagram of pin *C*.

As mentioned previously, the loading and the truss are symmetrical so that the forces in all the members are now determined. If the truss or loading were not symmetrical, however, the solution would be continued by proceeding to the next unmarked pin. This pin is *D*, but another pin having only two unmarked members acting upon it is pin *G*. It is preferable to avoid pin *D*, start anew from pin *G*, and determine the forces in *FG* and *EG*. After the action of *FG* and *EG* upon their end pins is indicated in the original truss diagram, the next pin to be selected for analysis is pin *F*. From the FBD of *F*, the forces *DF* and *EF* can be found. Next, the FBD of pin *E* will enable us to find the forces in *DE* and *CE*. The force in *CE* will then have been determined from the FBD at *C* and, again independently, from the FBD at *E*. A check on the accuracy of the work is thus obtained if the force in *CE* as found from pin *C* agrees with that found from pin *E*.

The final appearance of the original truss diagram after all the forces have been

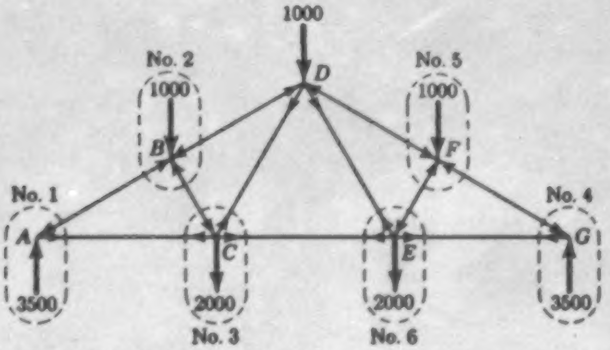


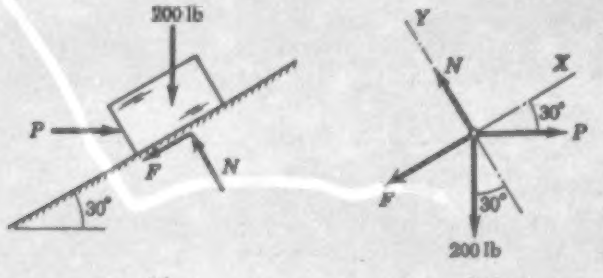
FIG. 4-12. — Order of taking free-body diagrams. All members marked indicates that all forces have been determined.

determined is shown in Fig. 4-12. This figure also indicates the order in which the free-body diagrams of the various pins would be drawn if the truss or the loading were not symmetrical.

Friction — Illustrative Problem

A 200-lb block is at rest on a 30° incline. The coefficient of friction between the block and the incline is 0.20. Compute the value of a horizontal force P that will cause motion to impend up the incline.

Solution: The FBD of the block is shown in Fig. 5-9a. Since motion is impending up the incline, the maximum static friction F is directed down the incline. A point diagram of the forces is formed by first selecting X and Y axes with the X axis parallel to and positive in the direction of impending motion, and then imagining the block squeezed to a point coincident with the origin of the axes. The forces on the body are then applied to this point to form the concurrent system shown in Fig. 5-9b. (*Note:* The point diagram is sometimes more convenient than the FBD for computing components.)



(a) (b)

FIG. 5-9.

The three unknowns N, F, and P are found from Eq. (5-1) and the two equations of equilibrium for concurrent forces. We now have

$$[\Sigma Y = 0] \qquad N - 200 \cos 30° - P \sin 30° = 0$$

$$N = 173.2 + 0.5\,P \tag{a}$$

$$[F = fN] \qquad F = 0.2(173.2 + 0.5\,P)$$

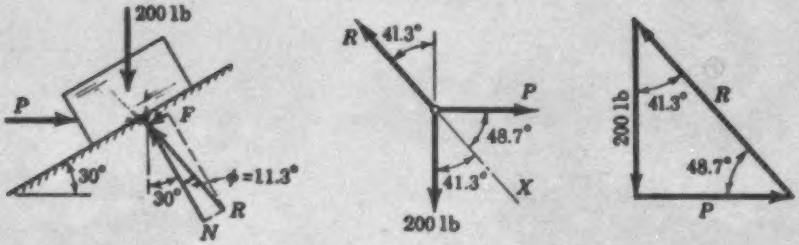
$$F = 34.64 + 0.1\,P \tag{b}$$

$$[\Sigma X = 0] \qquad P \cos 30° - 200 \sin 30° - F = 0$$

Substituting the value of F from (*b*) we obtain

$$P = 176 \text{ lb} \quad \textit{Ans.}$$

Resolve the problem below using the angle of friction ϕ and the total reaction of the incline on the block instead of its components F and N.



(a) (b) (c)

Fig. 5-10.

Solution: Whenever the normal pressure N must be expressed in terms of an unknown force, such as P in the preceding example, it will generally be simpler to use the total reaction R instead of its components F and N.

Since motion is impending, R will make the angle ϕ with N as shown in Fig. 5-10a. The value of ϕ is found from Eq. (5-2) to be

$$[\tan\phi = f] \qquad \tan\phi = 0.20 \qquad \phi = 11.3^\circ$$

The block is subjected to three forces in equilibrium. The point diagram of the forces acting on the block is shown in Fig. 5-10b. This system will be recognized as a concurrent force system in equilibrium and may be solved by the method developed in Art. 3-3.

If the X axis is taken through R, a Y summation (Y axis not shown) will determine P at once by eliminating R. We thereby obtain

$$[\Sigma Y = 0] \qquad P \sin 48.7^\circ - 200 \sin 41.3^\circ = 0 \qquad P = 176 \text{ lb} \quad \textit{Ans.}$$

A preferred variation of this solution when only three forces are involved consists of applying the sine law to the force polygon shown in Fig. 5-10c. Since equilibrium exists, the force polygon will close. The 200-lb weight is represented by the vertical vector shown. Through the tip of this vector, a horizontal line of indeterminate length is drawn to represent the known direction of P. From Fig. 5-10a, the known direction of R is $30^\circ + \phi = 41.3^\circ$ with the vertical. A line representing R may be drawn through the tail of the 200-lb vector to intersect P as shown.

Values may now be obtained graphically by scaling from the polygon, or analytically by applying the sine law. Using the latter, we have

$$\frac{P}{\sin 41.3^\circ} = \frac{200}{\sin 48.7^\circ}$$

whence as before

$$P = 176 \text{ lb} \quad \textit{Ans.}$$

Free Falling Bodies — Illustrative Problem

As shown in Fig. 10–4, a stone is thrown vertically into the air from a tower 100 ft high at the same instant that a second stone is thrown upward from the ground. The initial velocity of the first stone is 50 ft per sec and that of the second stone is 75 ft per sec. When and where will the stones be at the same height from the ground?

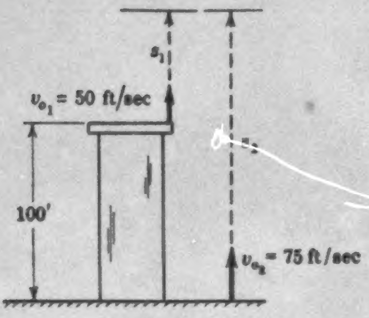


FIG. 10–4.

Solution: The initial direction of motion for each stone is upward. Using the convention established in Art. 10–2, we therefore take the upward direction as positive for s, v, and a. Applying Eq. (10–2) and noting that the acceleration is $g = 32.2$ ft per sec² directed downward and therefore negative, we obtain

$[s = v_0t + \frac{1}{2}at^2]$ For stone 1: $s_1 = 50t - 16.1t^2$ (a)

For stone 2: $s_2 = 75t - 16.1t^2$ (b)

From Fig. 10–4, $s_2 - s_1 = 100$. Hence subtracting Eq. (a) from Eq. (b) gives

$$s_2 - s_1 = 100 = 25t$$
$$t = 4 \text{ sec}$$

Substituting t in Eqs. (a) and (b), we have

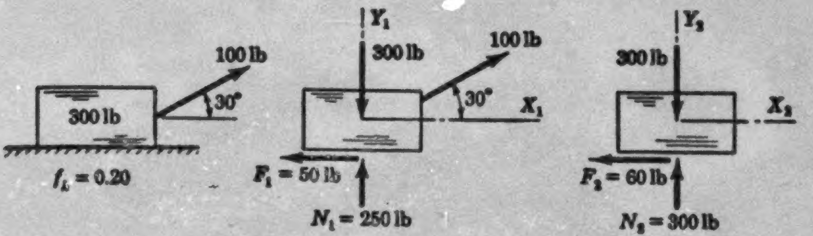
$$s_1 = 50 \times 4 - 16.1 \times (4)^2 = 200 - 257.6 \qquad s_1 = -57.6 \text{ ft}$$
$$s_2 = 75 \times 4 - 16.1 \times (4)^2 = 300 - 257.6 \qquad s_2 = 42.4 \text{ ft} \quad \textit{Ans.}$$

Hence the stones pass each other 57.6 ft below the top of the tower, or 42.4 ft from the ground. Note that although we assumed that they would pass above the tower, the negative sign of s_1 indicates otherwise. Since the terms involved in the equations are vector quantities, an incorrect assumption of direction results merely in a negative sign.

Work-Energy Method — Illustrative Problem

The 300-lb block in Fig. 14-4a rests upon a level plane for which the coefficient of kinetic friction is 0.20. Find the velocity of the block after it moves 80 ft, starting from rest. If the 100-lb force is then removed, how much farther will it travel?

Solution: The FBD of the block in its first phase of motion is shown in Fig. 14-4b. Computing the normal and frictional forces in the usual manner, we apply the work-energy equation to phase 1.



(a) (b) Phase 1 (c) Phase 2

FIG. 14-4.

$$\left[\Sigma X_1 \cdot s_1 = \frac{W}{2g}(v_1^2 - v_0^2)\right] \qquad (100 \cos 30^\circ - 50)(80) = \frac{300}{64.4} v_1^2$$

from which

$$v_1 = 25.1 \text{ ft per sec} \quad \textit{Ans.}$$

The FBD of the block during the second phase of the motion is shown in Fig. 14-4c. To determine how much farther the block will travel after the 100-lb force is removed, we equate the resultant work done during both phases of the motion to the total change in kinetic energy. This change in kinetic energy will be zero since the final and initial velocities are zero. We obtain

$$\left[\Sigma X_1 s_1 + \Sigma X_2 s_2 = \frac{W}{2g}(v_2^2 - v_0^2)\right] \qquad (100 \cos 30^\circ - 50)(80) - 60\, s_2 = 0$$

from which

$$s_2 = 48.7 \text{ ft} \quad \textit{Ans.}$$

Conservation of Momentum – Illustrative Problem

A ballistic pendulum consists of a sand box weighing 59 lb that is suspended from a cord 10 ft long. A 1-lb shell is fired horizontally into the box and remains embedded in it. Because of impact, the sand box swings through a maximum angle of 30°, as shown in Fig. 15-9. Determine the velocity with which the shell strikes the box.

Solution: The initial velocity of the sand box with the shell embedded is found by the work-energy method. The work done is negative because the gravity force acts opposite to the upward rise.

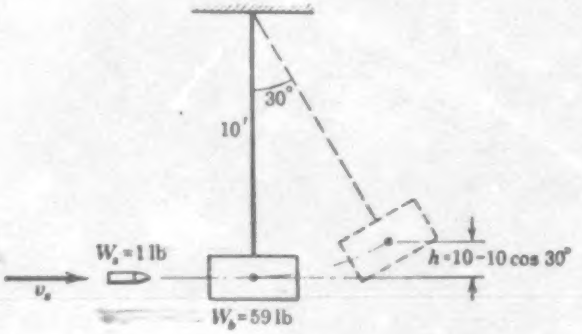


FIG. 15-9.

$$\left[-Wh = \frac{W}{2g}(v^2 - v_o^2)\right] \qquad -W(10 - 10\cos 30°) = \frac{W}{64.4}(0 - v_o^2)$$

$$v_o^2 = 64.4 \times 1.34 \qquad v_o = 9.3 \text{ ft per sec}$$

This value of velocity represents the common velocity of the shell and box directly after impact. The velocity of the shell before impact can be found by applying the principle of conservation of momentum.

$$\left[\frac{W_1 v_1}{g} + \frac{W_2 v_2}{g} = \frac{W_1 + W_2}{g} v\right] \qquad \frac{1 \times v_s}{g} + 0 = \frac{1 + 59}{g} \times 9.3$$

$$v_s = 558 \text{ ft per sec} \quad \textit{Ans.}$$

Strength of Materials

This branch of mechanics treats of the internal forces (stresses) in a physical body and of the changes of shape and size (strains) of the body, as well as their relation to the external forces (loads) that act on the body and the physical properties of the material of the body.

Illustrative problems are given below to show the application of algebra and geometry to the various types of problems encountered in the Strength of Materials course.

Centroids of Composite Areas – Sample Problem

SAMPLE PROBLEM Determine the location of the centroid of the plane figure shown in Fig. 10-11.

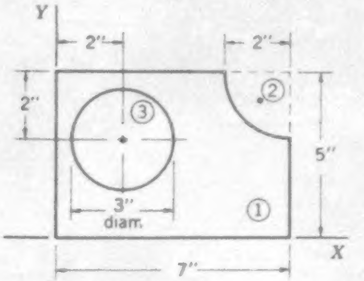


Figure 10-11 Diagram for Sample Problem 4.

Solution: Divide the composite figure into three simple areas, a rectangle 7 by 5, a quadrant with a 2-in. radius, and a circle with a 3-in. diameter. The rectangle is a positive area. The quadrant and hole are treated as negative areas.

$$A_1 = bh = 7(5) = 35.00 \text{ in.}^2$$

$$A_2 = \frac{-\pi d^2}{16} = \frac{-\pi(4)^2}{16} = -3.14 \text{ in.}^2$$

$$A_3 = \frac{-\pi d^2}{4} = \frac{-\pi(3)^2}{4} = \underline{-7.07} \text{ in.}^2$$

$$\Sigma A = 24.79 \text{ in.}^2$$

$$\bar{x} = \frac{\Sigma Ax}{\Sigma A} = \frac{(35)(3.50) - (3.14)(6.15) - (7.07)(2)}{24.79}$$

ΣAx is the sum of the products of each area and the distance from the y axis to the centroid of that area.

$$\bar{x} = \frac{122.5 - 19.3 - 14.14}{24.79} = \frac{89.06}{24.79} = 3.59 \text{ in.}$$

$$\bar{y} = \frac{\Sigma Ay}{\Sigma A} = \frac{(35)(2.5) - (3.14)(4.15) - (7.07)(3)}{24.79}$$

$$= \frac{87.5 - 13.05 - 21.21}{24.79} = \frac{53.24}{24.79} = 2.15 \text{ in.}$$

or

Area	Dimen.	A	x	Ax	y	Ay
1	7 × 5	35.00	3.50	122.5	2.50	87.5
2	$r = 2$	−3.14	6.15	−19.3	4.15	−13.05
3	$d = 3$	$\underline{-7.07}$	2.00	$\underline{-14.14}$	3.00	$\underline{-21.21}$
		$\Sigma A = 24.79$		$\Sigma Ax = 89.06$		$\Sigma Ay = 53.24$

$$\bar{x} = \frac{\Sigma Ax}{\Sigma A} = \frac{89.06}{24.79} = 3.59 \text{ in.}$$

$$\bar{y} = \frac{\Sigma Ay}{\Sigma A} = \frac{53.24}{24.79} = 2.15 \text{ in.}$$

Moment of Inertia of Composite Areas

SAMPLE PROBLEM Determine the moment of inertia of the area shown in Fig. 10-25:

(*a*) About the vertical gravity axis *YY*
(*b*) About a horizontal axis 1-1, 2 in. below the base

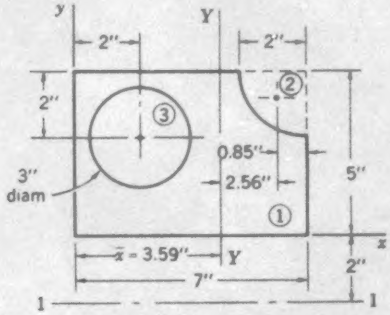


Figure 10-25 Diagram for Sample Problem 14*a*.

In calculating the moment of inertia of an area that has holes or cutouts in it, treat the areas and moments of inertia of these holes and cutouts as negative values.

Solution a: From Sample Problem 4, $\bar{x} = 3.59$ in.

$$I_1 = \frac{bh^3}{12} = \frac{5(7)^3}{12} = 143 \text{ in.}^4$$
$$I_2 = -0.055r^4 = -0.055(2)^4 = -0.88 \text{ in.}^4$$
$$I_3 = \frac{-\pi(d)^4}{64} = \frac{-\pi(3)^4}{64} = -3.98 \text{ in.}^4$$

I + Ad2 is the transfer theorem

d is distance from centroid of the part to x = 3.59

Area	Cent. (x)	I	A	d	d^2	Ad^2	$I + Ad^2$
1	3.50	143.0	35.0	0.09	0.0081	0.28	143.28
2	6.15	−0.88	−3.14	2.56	6.55	−20.6	−21.48
3	2.00	−3.98	−7.07	1.59	2.53	−17.9	−21.88
							$I_y =$ 99.92 in.^{4}
							say $I_y =$ 100 in.^{4}

Solution b: From Fig. 10·26.

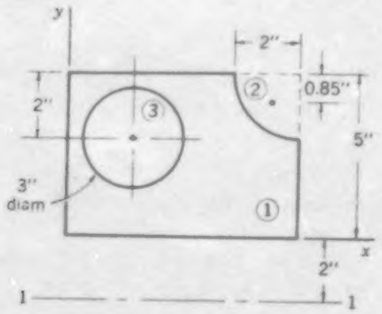


Figure 10·26 Diagram for Sample Problem

$$I_1 = \frac{bh^3}{12} = \frac{7(5)^3}{12} = 72.9 \text{ in.}^4$$

$$I_2 = -0.055r^4 = -0.055(2)^4 = -0.88 \text{ in.}^4$$

$$I_3 = \frac{-\pi d^4}{64} = \frac{-\pi(3)^4}{64} = -3.98 \text{ in.}^4$$

Area	I	A	d	d^2	Ad^2	$I + Ad^2$
1	72.9	35.0	4.50	20.25	710.0	782.9
2	−0.88	−3.14	6.15	37.80	−118.8	−119.68
3	−3.98	−7.07	5.00	25.00	−176.8	−180.78
						$I_{1-1} = 482.44$ in.^{4}
						say $I_{1-1} = 482$ in.^{4}

SAMPLE PROBLEM A solid shaft 8 in. in diameter has the same cross-sectional area as a hollow shaft of the same material, with inside diameter of 6 in.

(*a*) Compare the horsepower transmission of these shafts at the same rpm.
(*b*) Compare the angle of twist in equal lengths of these shafts when stressed to the same intensity.

Solution: Find the outside diameter d_o of the hollow shaft. Since the cross-sectional areas are equal,

$$\frac{\pi d^2}{4} = \frac{\pi}{4}(d_o^2 - d_i^2)$$
$$\frac{\pi(8^2)}{4} = \frac{\pi}{4}(d_o^2 - 6^2)$$
$$8^2 = d_o^2 - 6^2$$
$$d_o^2 = 64 + 36 = 100$$
$$d_o = 10 \text{ in.}$$

a. From equation for the solid shaft:

$$s_s = \frac{16T}{\pi d^3} \quad \text{or} \quad T = \frac{\pi d^3 s_s}{16}$$
$$T = \frac{\pi(8^3)s_s}{16}$$

From Eq. (13·7) for the solid shaft,

$$\text{hp}_{\text{solid}} = \frac{Tn}{63{,}000} = \frac{\pi(8^3)s_s n}{63{,}000(16)} = \frac{\pi s_s n}{(63{,}000)16}(512)$$

From Eq. (13·4) for the hollow shaft,

$$s_s = \frac{16Td_o}{\pi(d_o^4 - d_i^4)} \quad \text{or} \quad T = \frac{\pi(d_o^4 - d_i^4)s_s}{16d_o}$$
$$T = \frac{\pi(10^4 - 6^4)s_s}{16(10)}$$

SAMPLE PROBLEM A run of 4-in. schedule 40 seamless steel pipe (4.50 in. OD, 0.237 in. wall thickness) is to carry a ¼-ton-capacity chain hoist attached midway between pipe support hangers. The ultimate tensile strength of the steel pipe is 48,000 psi. A safety factor of 4 is specified. The pipe weighs 10 lb per ft. Assume no additional stress or load due to internal pressure. Treat the length of pipe between hangers as a simply supported beam. Find the maximum safe spacing of pipe support hangers.

Solution: Sketch the system as in Fig. 12·4.

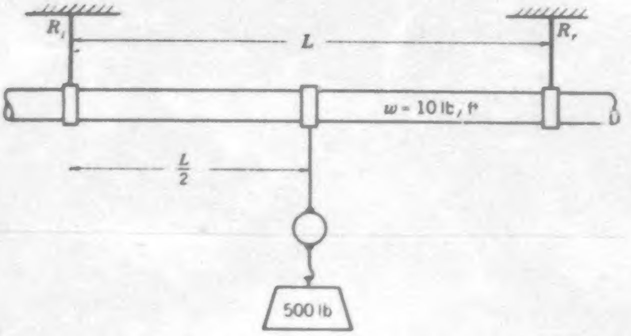


Figure 12·4 Diagram for Sample Problem 2.

$$\Sigma F_v = 0$$

$$R_l + R_r - 10L - 500 = 0$$
$$R_l = R_r = 5L + 250 \text{ (due to symmetry)}$$

Find M_{max} at center of span.

$$\begin{aligned}
M_{max} &= R_l\left(\frac{L}{2}\right) - \frac{10L}{2}\left(\frac{L}{4}\right)\\
&= (5L + 250)\,\frac{L}{2} - \frac{5L^2}{4}\\
&= \frac{5L^2}{2} + 125L - \frac{5L^2}{4} = \frac{5L^2}{4} + 125L \text{ ft-lb}\\
&= 15L^2 + 1{,}500L \text{ in.-lb}
\end{aligned}$$

For a hollow circular section,

$$\begin{aligned}
I &= \frac{\pi}{64}(d_o^4 - d_i^4) = \frac{\pi}{64}(4.50^4 - 4.03^4)\\
&= \frac{\pi}{64}(410 - 262) = \frac{\pi}{64}(148) = 7.26 \text{ in.}^4\\
c &= \frac{4.50}{2} = 2.25 \text{ in.}
\end{aligned}$$

Allowable stress = 48,000/4 = 12,000 psi. From Eq. (12·1),

$$s = \frac{Mc}{I} \qquad M = \frac{sI}{c}$$
$$M = \frac{12{,}000(7.26)}{2.25} = 38{,}600 \text{ in.-lb}$$
$$15L^2 + 1{,}500L = 38{,}600$$
$$L^2 + 100L - 2{,}570 = 0$$

This quadratic equation may be solved by quadratic formula.

$$\begin{aligned}
L &= \frac{-100 \pm \sqrt{100^2 - 4(1)(-2{,}570)}}{2} = \frac{-100 \pm \sqrt{10{,}000 + 10{,}280}}{2}\\
&= \frac{-100 \pm \sqrt{20{,}280}}{2} = \frac{-100 \pm 143}{2}
\end{aligned}$$

Selecting the positive result, we obtain

$$L = \frac{-100 + 143}{2} = \frac{43}{2} = 21.5 \text{ ft (maximum safe spacing of supports)}$$

Member Composed of Two Different Materials in Series

SAMPLE PROBLEM A steel bar, 1 in. square and 8 in. long, is set end to end with a C.I. (cast iron) Class 40 bar, 2 in. square and 4 in. long, between two immovable supports. What stress will develop in each material due to a temperature rise of 50°F?

Solution: From Appendix B, Table 1,

$$E_s = 30 \times 10^6 \text{ psi (for steel)}$$
$$E_c = 16 \times 10^6 \text{ psi (for C.I. Class 40)}$$

From Table 8·2,

$$\alpha_s = 6.5 \times 10^{-6} \text{ in. per in. per °F (for steel)}$$
$$\alpha_c = 6.3 \times 10^{-6} \text{ in. per in. per °F (for C.I.)}$$

If the member were not restrained, each material would elongate by amounts of

$$\delta_s = \alpha_s l_s \, \Delta t = 6.5 \times 10^{-6} (8)(50) = 0.00260 \text{ in.}$$
$$\delta_c = \alpha_c l_c \, \Delta t = 6.3 \times 10^{-6} (4)(50) = 0.00126 \text{ in.}$$

The total elongation, δ, that would occur would equal $\delta = \delta_s + \delta_c = 0.00260 + 0.00126 = 0.00386$ in.

Since the member is not free to elongate, the materials are placed in compression by a force F.

$$F = \frac{\delta}{\dfrac{\ell_s}{A_s E_s} + \dfrac{\ell_c}{A_c E_c}} = \frac{0.00386}{\dfrac{8}{(1)(30 \times 10^6)} + \dfrac{4}{(4)(16 \times 10^6)}}$$

$$F = \frac{0.00386}{0.27 \times 10^{-6} + 0.06 \times 10^{-6}} = \frac{0.00386}{0.33 \times 10^{-6}} = 11{,}600 \text{ lb}$$

$$S_s = \frac{F}{A_s} = \frac{11{,}600}{1} = 11{,}600 \text{ psi (compression in steel)}$$

$$S_c = \frac{F}{A_c} = \frac{11{,}600}{4} = 2{,}900 \text{ psi (compression in cast iron)}$$

SAMPLE PROBLEM A simply supported, 12 WF 65 beam, 16 ft long, carries concentrated loads of 6,000 lb at each quarter point and is subjected to an axial tensile force of 25,000 lb applied at the end sections.

(*a*) Find maximum combined tensile stress and maximum combined compressive stress.

(*b*) If it were necessary to make a 1½-in. hole in the web of this beam at the center cross section so that a water pipe can be accommodated, where on this cross section would you recommend that the hole center be located?

Solution a: The weight of the beam is 65(16) = 1,040 lb. The total vertical load on the beam is 6,000 + 6,000 + 6,000 = 18,000 lb. Since the weight of the beam is only (1,040/18,000)100 = 5.8 percent of the total vertical load, it may be neglected without excessive error (approximately 4 percent error occurs here).

Figure 14-6 shows the beam with its shear-force and bending-moment diagrams. Note that the 25,000-lb axial force does not affect these diagrams. For the 12 WF 65, $A = 19.11$ in.^{2}, $Z = 88.0$ in.^{3}, and $d = 12.12$ in. (Appendix B, Table 4).

Direct stress:

$$s_1 = \frac{F}{A} = \frac{25{,}000}{19.11} = 1{,}310 \text{ psi (tension)}$$

Bending stress:

$$s_2 = \frac{Mc}{I} = \frac{M}{Z}$$

The shear diagram indicates that the center cross section is the location of the maximum moment. A free-body diagram of the left half of the beam facilitates calculating $M_{\max}$ (Fig. 14-7).

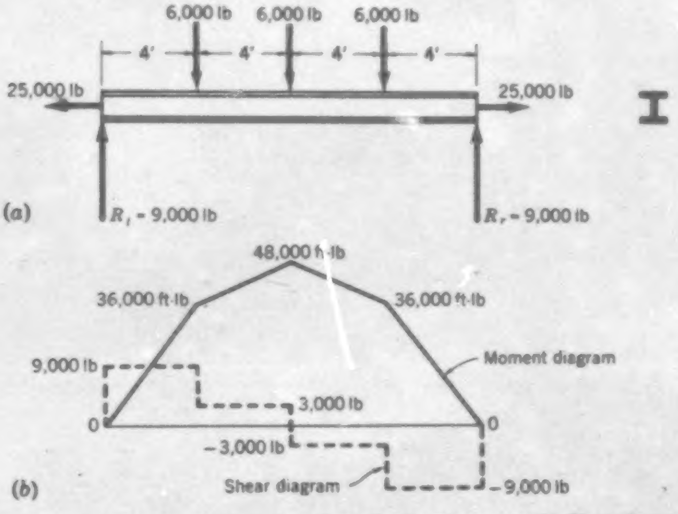

6,000 lb
6,000 lb
6,000 lb
4'
4'
4'
4'
25,000 lb
25,000 lb
(a)
R₁ = 9,000 lb
R₂ = 9,000 lb
48,000 ft-lb
36,000 ft-lb
36,000 ft-lb
9,000 lb
Moment diagram
3,000 lb
0
0
−3,000 lb
(b)
Shear diagram
−9,000 lb

Figure 14-6 (*a*) Beam diagram for Sample Problem (*b*) Shear-force and bending-moment diagrams.

$\Sigma M = 0$ about an axis through the center cross section gives

$$M_{\max} = 9{,}000(8) - 6{,}000(4) = 72{,}000 - 24{,}000$$
$$= 48{,}000 \text{ ft-lb}$$
$$s_2 = \frac{M}{Z} = \frac{48{,}000(12)}{88.0} = 6{,}550 \text{ psi}$$
$$s_2 = 6{,}550 \text{ psi} \begin{cases} \text{tension at bottom fiber} \\ \text{compression at top fiber} \end{cases}$$

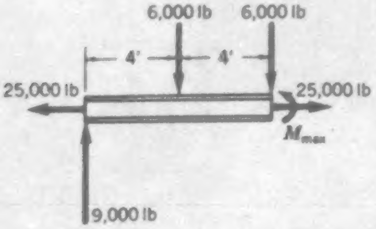


Figure 14-7 Free-body diagram of left half of beam for Sample Problem

Therefore, at center cross section,

$$\text{Top fiber } s = s_2 - s_1 = 6{,}550 - 1{,}310$$
$$\text{Top fiber } s = 5{,}240 \text{ psi (maximum compression)}$$
$$\text{Bottom fiber } s = s_2 + s_1 = 6{,}550 + 1{,}310$$
$$\text{Bottom fiber } s = 7{,}860 \text{ psi (maximum tension)}$$

Solution b: The distribution of combined stress at the center cross section is shown in Fig. 14-8.

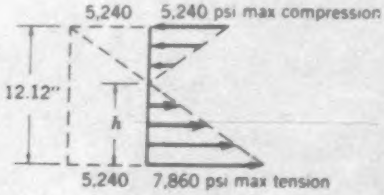


Figure 14-8 Location of point of zero stress from distribution of combined stress at center cross section.

The location of zero combined stress would be the most preferable position for a hole in the web. The level of zero stress can be found from Fig. 14-8 by similar triangles.

$$h = \left(\frac{7{,}860}{7{,}860 + 5{,}240}\right) 12.12 = \left(\frac{7{,}860}{13{,}100}\right) 12.12 = 7.27 \text{ in.}$$

The center of the $1\frac{1}{2}$-in. hole should be located 7.27 in. above the bottom of the lower flange.

Fluid Mechanics

The course in fluid mechanics covers the study and behavior of fluids (liquids and gases) in industrial systems, particularly pipeline and duct systems, and fluid machinery such as pumps and fans.

Applying the basic conservation equations (mass, energy and momentum), hydraulic principles and fluid properties to industrial systems enables one to predict their performance and establish design parameters.

The laboratory experiments reinforce the theory and provide the practical experience in measuring fluid properties related to the design and operation of industrial systems.

Problems in fluid mechanics involve the use of certain basic equations, the manipulation of them to solve for certain unknowns (literal equations), and the conversion of units. The latter can often be accomplished by manipulating units as if they were algebraic terms.

Important equations are:

Bernoulli's Equation which is basically an energy equation:

$$Z_1 + \frac{P_1}{\gamma} + \frac{v_1{}^2}{2g} = Z_2 + \frac{P_2}{\gamma}\ \frac{v_2{}^2}{2g}$$

Continuity Equation: The quantity of liquid that flows past any point per unit time remains constant.

$$Q = A_1\, v_1 = A_2\, v_2$$

Both equations assume incompressability (i.e. density is constant). In these equations, Z is height, P is pressure, v is velocity, γ is density in $\frac{\text{lbs}}{\text{ft}^3}$, ρ is density in $\frac{\text{slugs}}{\text{ft}^3}$.

Conversion from γ to ρ is obtained by dividing 32 $\frac{\text{ft}}{\text{sec}^2}$, i.e.,

$$\rho = \gamma \div 32\,\frac{\text{ft}}{\text{sec}^2}\quad \frac{\text{lbs}}{\text{ft}^3} \div \frac{\text{ft}}{\text{sec}^2} = \frac{\text{lb-sec}^2}{\text{ft}^4} = \frac{\text{slugs}}{\text{ft}^3}$$

For example:

1. What is the weight of 25 gallons of gasoline: 7.480 gal = 1 ft,

$$R = 1.32\,\frac{\text{slugs}}{\text{ft}^3}\ .$$

$$w = \gamma V = 32\,\frac{\text{ft}}{\text{sec}^2} \times \frac{1.32\ \text{lb-sec}^2}{\text{ft}^4} \times 25\ \text{gal} \times \frac{1\ \text{ft}^3}{7.480\ \text{gal}} = 142\ \text{lbs.}$$

2. Specific gravity of a liquid, S.G.= $\frac{\rho\ \text{liquid}}{\rho\ \text{water}} = \frac{\gamma\ \text{liquid}}{\gamma\ \text{water}}$

How far will a block sink into a fluid? A block will sink until it displaces its own weight of water. If a block has length 50 ft. and width 20 ft., it will sink a depth of D ft. If the block weighs 75 tons or 150,000 lbs., it will displace a block of water of dimensions 20 ft. by 50 ft. by D ft., weighing 62.4 $\frac{\text{lbs}}{\text{ft}^3}$.

$$\text{Hence } 150{,}000 \text{ lbs.} = 62.4 \frac{\text{lb}}{\text{ft}^3} \times 20 \text{ ft} \times 50 \text{ ft} \times \text{D ft}$$

$$D = \frac{150{,}000 \text{ lbs ft}^3}{(62.4 \text{ lbs})(20 \text{ ft})(50 \text{ ft})} = 2.40 \text{ ft.}$$

Fluid Properties

One gallon of a certain fuel oil weighs 7.50 pounds. Determine its specific weight (γ), its density (ρ) and its specific gravity (SG).

$V = 1$ Gal $\qquad\qquad W = 7.5$ lbs

$$V = 1 \text{ gal} \times \frac{1 \text{ ft}^3}{7.48 \text{ gal}} = .134 \text{ ft}^3$$

$$\gamma = \frac{W}{V} = \frac{7.5}{.134} \qquad \boxed{\gamma = \frac{56.1 \text{ lbs}}{\text{ft}^3}}$$

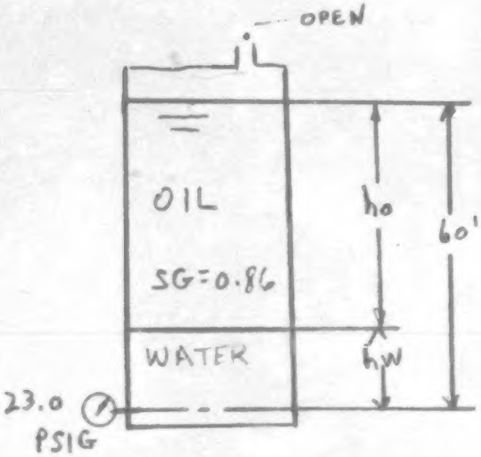
$$\rho = \frac{\gamma}{g} = \frac{56}{32.2} \qquad \boxed{\rho = \frac{1.74 \text{ slugs}}{\text{ft}^3}}$$

$$SG = \frac{\gamma}{62.4} = \frac{\rho}{1.94}$$

$$= \frac{56}{62.4} = \frac{1.74}{1.94} \qquad \boxed{SG = 0.90}$$

Pressure

An oil storage tank is open to the atmosphere as shown. Some water was accidentally pumped into the tank and settled to the bottom. If a pressure gage at the bottom reads 23.0 psig, determine the depths of the oil and water.



$$P\text{ bottom} = 0 + \gamma_o h_o + \gamma_w h_w$$

$$23.0 \frac{\text{lbs}}{\text{in}^2} \times 144 \frac{\text{in}^2}{\text{ft}^2} = .86 \times 62.4\, h_o + 62.4\, h_w$$

$$h_o = 60 - h_w$$

$$\frac{23 \times 144}{62.4} = \frac{.86 \times 62.4\,(60 - h_w) + 62.4\, h_w}{62.4}$$

$$53.1 = 51.6 - .86 h_w + h_w$$

$$1.5 = .14\, h_w$$

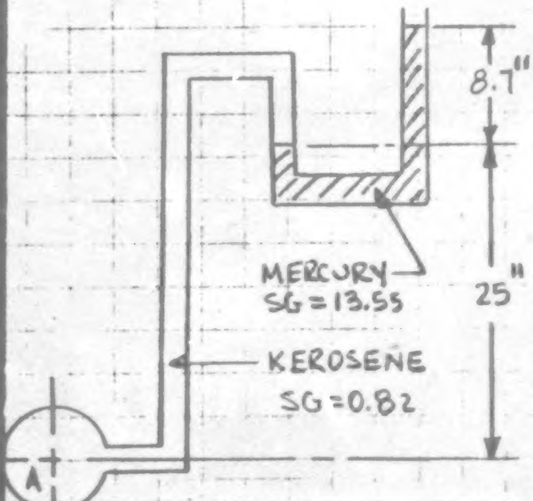
$$h_w = \frac{1.5}{.14}$$

$$h_o = 60 - 10.7$$

$$\boxed{\begin{aligned} h_w &= 10.7 \text{ ft} \\ h_o &= 49.3 \text{ ft} \end{aligned}}$$

Manometers

A manometer, open on one end, is connected to a pipe in which kerosene is flowing. If the difference in the fluid levels in the tube is as shown, determine the gage pressure in the pipe at point A.



$$P_A = 0 + \gamma_m h_m + \gamma_k h_k$$

$$P_A = 13.55 \times 62.4 \frac{\text{lbs}}{\text{ft}^3} \times 8.7 \text{ in} \times \frac{1 \text{ ft}^3}{1728 \text{ in}^3} + .82 \times 62.4 \times \frac{25}{1728}$$

$$P_A = \frac{62.4}{1728} \left(\overbrace{13.55 \times 8.7}^{117.9} + \overbrace{.82 \times 25}^{20.5} \right)$$

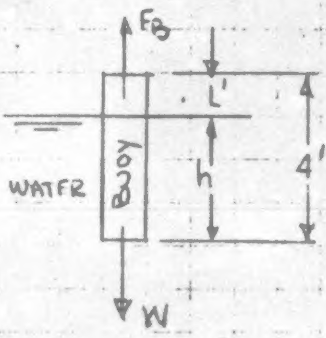
$$P_A = \frac{62.4}{1728} \times 138.4$$

$\boxed{P_A = 5.0 \text{ PSIG}}$

Buoyant Forces

A buoy is a solid cylinder, 1.0 foot in diameter and 4 feet long. It is made of a material with a specific weight of $\frac{50 \text{ lbs}}{\text{cubic ft}}$. If it floats upright, how much of its length is above the water?

$$D = 1.0 \text{ ft} \qquad L = 4 \text{ ft} \qquad \gamma_B = \frac{50 \text{ lbs}}{\text{ft}^3}$$



$$\Sigma F = 0: \qquad F_B = W$$

$$W = \gamma_B V_B = 50 \text{ x } \frac{\pi + 1^2}{\cancel{4}} \text{ x} \cancel{4} = 157 \text{ lbs.}$$

$$F_B = \gamma_W V_{Displ} = 62.4 \text{ x } \frac{\pi + 1^2}{4} \text{ x } h = 49h$$

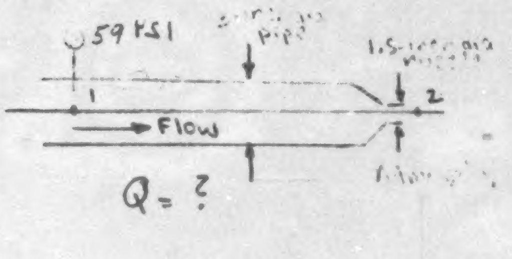
$$F_B = W$$

$$49h = 157$$

$$h = \frac{157}{49} = 3.20 \text{ ft.} \qquad \therefore \boxed{L^1 = 0.80 \text{ ft.}}$$

Bernoulli Equation + Ideal Flow

Determine the rate of flow of gasoline (SG = 0.72), in gpm, through the horizontal pipe shown. Neglect losses.



Gasoline: SG = .72

$Z_1 = Z_2 = 0 \quad P_1 = 59\ \text{PSI} \quad P_2 = 0\ \text{PSI}$

$D_1 = 3\ \text{in} \qquad A_1 = .0491\ \text{ft}^2$

$D_2 = 1.5\ \text{in.} \quad A_2 = .01227\ \text{ft}^2$

Bernoulli: $$Z_1 + \frac{v_1^{\,2}}{2g} + \frac{P_1}{\gamma} = Z_2 + \frac{v_2^{\,2}}{2g} + \frac{P_2}{\gamma}$$

$$0 + \frac{v_1^{\,2}}{64.4} + \frac{59\ \frac{\text{lbs}}{\text{in}^2} \times 144\ \frac{\text{in}^2}{\text{ft}^2}}{.72 \times 62.4\ \frac{\text{lbs}}{\text{ft}^3}} = 0 + \frac{v_2^{\,2}}{64.4} + 0$$

$$v_2^{\,2} - v_1^{\,2} = \frac{64.4 \times 59 \times 144}{.72 \times 62.4} = 12180$$

Continuity: $A_1 v_1 = A_2 v_2$

$$v_2 = \frac{A_1 v_1}{A_2} = \frac{.0491 \times v_1}{.01227} = 4v_1$$

$$(4v_1)^2 - v_1^{\,2} = 12180$$

$$16v_1^{\,2} - v_1^{\,2} = 12180 = 15v_1^{\,2}$$

$$v_1 = \left(\frac{12180}{15}\right)^{1/2} = (812)^{1/2} = 28.5\ \frac{\text{ft}}{\text{sec}}$$

$$Q = A_1 v_1 = .0491\ \text{ft}^2 \times 28.5\ \frac{\text{ft}}{\text{sec}} \times \frac{449\ \text{GPM}}{1\ \text{CFS}}$$

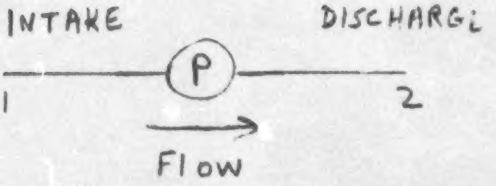
$$Q = .491 \times 2.85 \times .449 \times 10^3$$

$$\therefore \boxed{Q = 628\ \text{GPM}}$$

Pump Performance

The diameter of the discharge pipe of a pump is 6 inches and that of the intake pipe is 8 inches. A gage at discharge indicates a pressure of 30 PSI and a vacuum gage at intake reads 5 PSI. If water flows at 3 CFS and the brake horsepower is 35, determine the pump efficiency. Assume $\Delta Z = 0$.

Solution:



$Z_1 = Z_2$

$D_1 = 8$ in $A_1 = .3491$ ft^2

$P_1 = -5$ PSI

$D_2 = 6$ in $A_2 = .1964$ ft^2

$P_2 = 30$ PSI

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} + h_p = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma}$$

$$h_p = \frac{P_2 - P_1}{\gamma} + \frac{v_2^2 - v_1^2}{2g}$$

$$\frac{P_2 - P_1}{\gamma} = \frac{(30\,\frac{\text{lbs}}{\text{in}^2} - 5\,\frac{\text{lbs}}{\text{in}^2})}{62.4\,\frac{\text{lbs}}{\text{ft}^3}} \times \frac{144\text{ in}^2}{1\text{ ft}^2} = \frac{35 \times 144}{62.4} = 80.8\text{ ft}$$

$$v_2 = \frac{Q}{A_2} = \frac{3}{.1964} = 15.3\,\frac{\text{ft}}{\text{sec}}$$

$$v_1 = \frac{Q}{A_1} = \frac{3}{.3491} = 8.6\,\frac{\text{ft}}{\text{sec}}$$

$$\frac{v_2^2 - v_1^2}{2g} = \frac{15.3^2 - 8.6^2}{64.4} = \frac{234 - 74}{64.4} = 2.49\text{ ft.}$$

$$h_p = 80.8 + 2.49 = 83.3\text{ ft}$$

$$\text{FHP} = \frac{\gamma QH}{550} = \frac{62.4 \times 3 \times 83.3}{550} = 28.4$$

$$e = \frac{\text{FHP}}{\text{BHP}} = \frac{28.4}{35.0} = .811$$

$$\boxed{e = 81.1\,\%}$$

Pump Performance

A centrifugal pump discharged 300 gpm against a head of 55 feet when the rotative speed was 1500 rpm. The diameter of the impeller was 12.5 inches and the brake horsepower required was 6.0. A similar pump, 15.0 inches in diameter, is to run at 1750 rpm. Assuming equal efficiencies, determine:

(a) the head developed by the 15-inch pump;
(b) the rate of flow through the 15-inch pump;
(c) the brake horsepower required to drive the 15-inch pump.

Pump No. 1	Pump No. 2
Q = 300 GPM H = 55 ft	Q = ? H = ?
N = 1500 RPM D = 12.5 in.	N = 1750 RPM D = 15.0 in
BHP = 6.0	BHP = ?

$$\frac{H_1}{N_1^2\,D_1^2} = \frac{H_2}{N_2^2\,D_2^2}$$

$$H_2 = H_1 \left[\frac{N_2}{N_1} \times \frac{D_2}{D_1}\right]^2 = 55 \times \left[\frac{1750}{1500} \times \frac{15}{12.5}\right]^2$$

$$\boxed{H_2 = 108 \text{ ft}}$$

$$\frac{Q_1}{N_1\,D_1^3} = \frac{Q_2}{N_2\,D_2^3}$$

$$Q_2 = Q_1 \times \frac{N_2}{N_1} \times \left(\frac{D_2}{D_1}\right)^3 = 300 \times \frac{1750}{1500} \times \left(\frac{15}{12.5}\right)^3$$

$$\boxed{Q_2 = 605 \text{ GPM}}$$

$$e_1 = e_2$$

$$e_1 = \frac{FHP_1}{BHP_1} = \frac{\gamma Q_1 H_1}{550\ BHP_1} \qquad e_2 = \frac{\gamma Q_2 H_2}{550\ BHP_2}$$

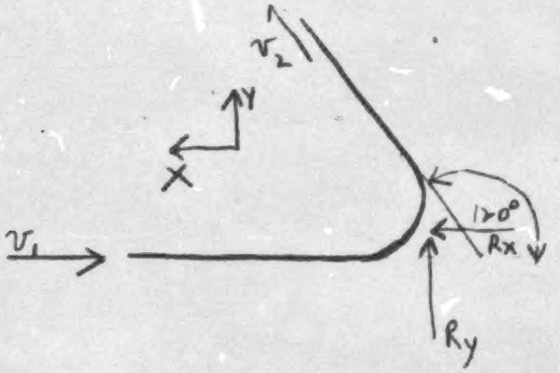
$$\frac{\gamma Q_1 H_1}{550\ BHP_1} = \frac{\gamma Q_2 H_2}{550\ BHP_2}$$

$$BHP_2 = BHP_1 \times \frac{Q_2}{Q_1} \times \frac{H_2}{H_1} = 6 \times \frac{605}{300} \times \frac{108}{55}$$

$$\boxed{BHP_2 = 23.8}$$

Forces Due to Fluid in Motion

A 2-inch diameter jet of water strikes a vane making an angle of 120^o with the direction of the jet. The jet strikes the vane with an initial velocity of 100 ft/sec but the velocity leaving the vane is reduced to 80 ft/sec. Determine the magnitude of the resultant force required to hold the vane in place.



D = 2-in

D = 2 in $A = .0218\ ft^2$

$v_1 = 100 \dfrac{ft}{sec}$ $v_2 = 80 \dfrac{ft}{sec}$

$R_x = ?$ $R_y = ?$

$\rho = 1.94 \dfrac{slugs}{ft^3}$

$F_x = \rho\, Q\, (v_{2x} - v_{1x}) = R_x$ $v_{1x} = -100$ $v_{1y} = 0$

$F_y = \rho\, Q\, (v_{2y} - v_{1y}) = R_y$ $v_{2x} = +80 \text{ x } \cos 60^o = +40$

$v_{2y} = +80 \text{ x } \sin 60^o = 69$

$Q = A_1 \text{ x } v_1 = \dfrac{\pi}{4} \text{ x } \left(\dfrac{2}{12}\right)^2 \text{ x } 100 = 2.18 \text{ CFS}$

4 23

$R_x = 1.94 \text{ x } 2.18\ (+40 + 100) = 592 \text{ lbs}$

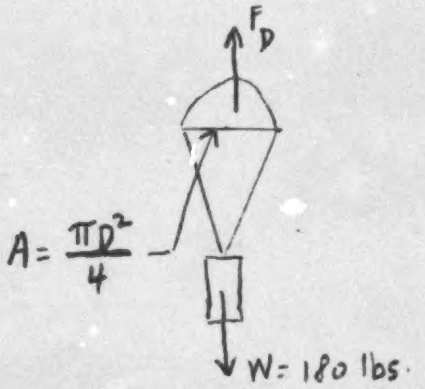
$R_y = 1.94 \text{ x } 2.18 \text{ x } (69 - 0) = 292 \text{ lbs}$

$R = (R_x^2 + R_y^2)^{1/2}$

$R = (592^2 + 292^2)^{1/2}$ $\boxed{R = 660 \text{ lbs}}$

Drag Forces

Calculate the required diameter of a parachute supporting a man weighing 180 pounds if the terminal velocity in air at 100° F is to be 15 ft/sec.



$v_T = 15 \frac{\text{ft}}{\text{sec}}$ Air Temp = 100° F

$\rho = 2.20 \times 10^{-3} \frac{\text{slugs}}{\text{ft}^3}$

Hemispherical cup $C_D = 1.35$

At terminal velocity, $\Sigma F = 0$

$W = F_D = \frac{1}{2} C_D \rho A_v{}^2$

$180 = \frac{1}{2} \times 1.35 \times 2.20 \times 10^{-3} \times \frac{\pi D^2}{4} \times (15)^2$

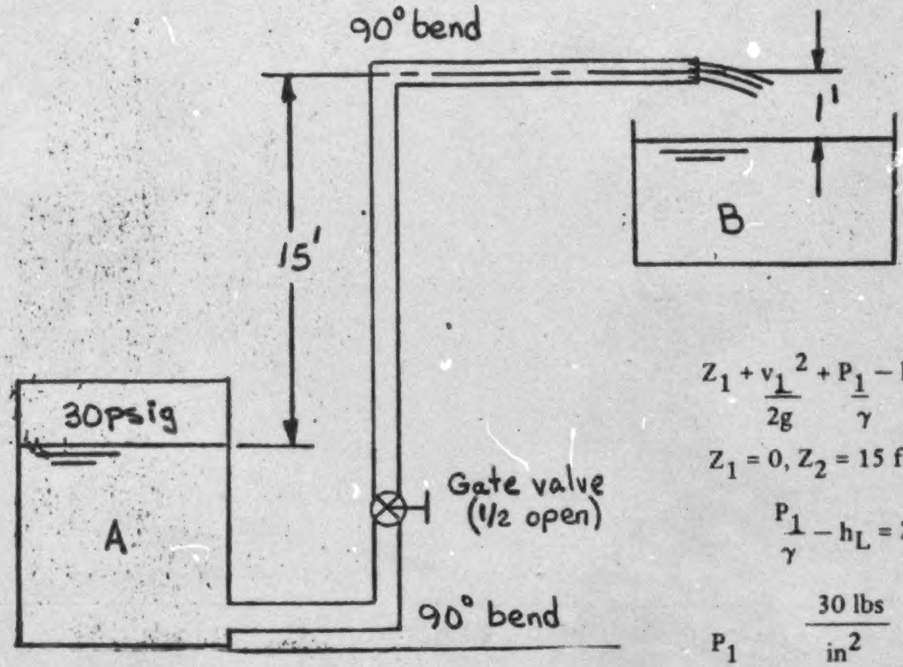
$180 = .262\ D^2$ $D^2 = \frac{180}{.262}$

$\boxed{D = 26.2 \text{ feet}}$

Neglect F_B
and W_{chute}

Energy Equation: Losses + Additions

Turpentine at 77° F is flowing in the system shown. The total length of 2-inch Type K copper tubing is 100 feet. The 90° bends have a radius of 12 inches. Determine the rate of flow into Tank "B" in gpm if a pressure of 30 psig is maintained above the turpentine in Tank "A".



Turpentine @ 77°F

$\mu = 2.87 \times 10^{-5}$ lb - sec/ft^2

$\gamma = 54.2$ lb/ft^3 $\quad \rho = 1.69$ slugs/ft^3

Type K Copper Tubing: L = 100 ft

D = 2 in. nominal

D = 0.1633 ft $\quad$ A = .02093 ft^2

D = 1.96 in $\quad \varepsilon/D = .00003$

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} - h_L = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma}$$

$Z_1 = 0$, $Z_2 = 15$ ft, $v_1 = 0$, $v_2 = v$, $P_1 = 30$ PSI, $P_2 =$

$$\frac{P_1}{\gamma} - h_L = Z_2 + \frac{v^2}{2g}$$

$$\frac{P_1}{\gamma} = \frac{\frac{30 \text{ lbs}}{\text{in}^2} \times \frac{144 \text{ in}^2}{\text{ft}^2}}{\frac{54.2 \text{ lbs}}{\text{ft}^3}} = 80 \text{ ft of turpentine}$$

$$80 - h_L = 15 + \frac{v^2}{2g}$$

$$h_L + \frac{v^2}{2g} = 65 \text{ Energy Equation}$$

$$h_L = h_1 + h_2 + h_3 + h_4$$

$h_1 = f \dfrac{L}{D} \dfrac{v^2}{2g}$ $\quad$ Friction $\quad$ $\dfrac{L}{D} = \dfrac{100}{1633} = 613$

$h_2 = K \dfrac{v^2}{2g}$ $\quad$ Entrance $\quad$ K = 0.50

$h_3 = f \dfrac{L_e}{D} \dfrac{v^2}{2g}$ $\quad$ Gate Valve ½ open $\quad$ $\dfrac{L_e}{D} = 160$

$h_4 = f \dfrac{L_e}{D} \dfrac{v^2}{2g}$ $\quad$ 2 – 90° Bends

$$\frac{r}{D} = \frac{1 \text{ ft}}{.1633 \text{ ft}} = 6.13 \qquad \frac{L_e}{D} = 2 \times 18 = 36$$

$$\Sigma\frac{L}{D} = 613 + 160 + 36 = 809 \qquad \Sigma K = .5$$

$$\therefore h_L = \frac{v^2}{2g}\left(f\Sigma\frac{L}{D} + \Sigma K\right) = \frac{v^2}{2g}(809 \times f + .5)$$

Energy Equation: $\frac{v^2}{2g}(809 \times f + 1.5) = 65$

$$v^2 = \frac{65 \times 64.4}{809f + 1.5}$$

$$v = \left[\frac{4186}{809f + 1.5}\right]^{1/2} \qquad \frac{\epsilon}{D} = .00003$$

Trial No. 1: $f = .0165$ $\quad 809f + 1.5 = 13.3 + 1.5 = 14.8$

$$v = \left[\frac{4186}{14.8}\right]^{1/2} = \left[283\right]^{1/2} = 16.8\ \frac{\text{ft}}{\text{sec}}$$

$$R_N = \frac{v \times D \times \rho}{\mu} = \frac{16.8 \times .1633 \times 1.69}{2.87 \times 10^{-5}} = 1.62 \times 10^5$$

$f = .0165$

f unchanged $\therefore v = 16.8$ ft/sec

$$\boxed{Q = 160 \text{ GPM}}$$

Thermodynamics

This course is a presentation of the fundamental concepts of thermodynamics, application of the various laws of thermodynamics, calculations based on the various ideal cycles and practical experience in determining the operating characteristics of many thermodynamic devices.

The presentation of the fundamental concepts includes length, area, volume and time; velocity, acceleration, mass, force and weight; density, specific volume, pressure and temperature; potential, kinetic, internal energy, heat and work; specific heat, enthalpy and entropy; molecular weight and the gas constants.

In addition, the laws of thermodynamics which are considered include the Zeroth Law, the First Law, the Perfect Gas Law, the Law of Conservation of Mass, the Second Law, Ideal Gas Process Equations.

Example

A 9-inch diameter piston-cylinder contains a gas which, under constant pressure, extends the piston 3 inches. Determine the work of this process if the gas pressure is 85 lbf/in² absolute.

Solution

In this case the pressure is constant so that the equation can be written

$$
\begin{aligned}
Wk_{es} &= p\int_{V_1}^{V_2} dV \\
&= p(V_2 - V_1) \\
&= p\Delta V
\end{aligned}
$$

The pressure is given as 85 psia and the change in volume can be calculated from

$$
\begin{aligned}
\Delta V &= A \times 3 \text{ in} \\
&= \pi r^2 \times 3 \text{ in} \\
&= 3.14 \times 4.5^2 \times 3 \text{ in}^3 = 190.8 \text{ in}^3
\end{aligned}
$$

which gives us

$$
\begin{aligned}
Wk_{es} &= 85 \text{ lbf/in}^2 \times 190.8 \text{ in}^3 \\
&= 16214.2 \text{ in-lbf} = 1351.2 \text{ ft-lbf}
\end{aligned}
$$

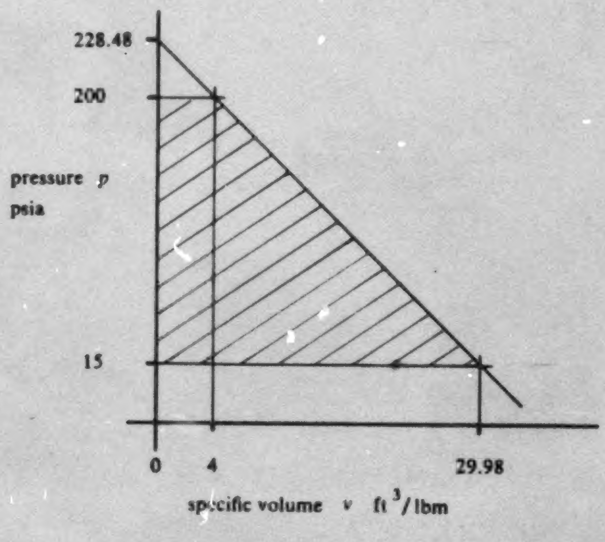
Answer

Example

A steam turbine running under reversible steady flow conditions takes in steam at 200 psia and exhausts it at 14 psia. Assuming the steam has a specific volume of 4.0 ft³/lbm at the inlet and the pressure-volume relation is

$$p = 228.48 - 7.12v$$

where p is in psia units and v is in ft³/lbm units, determine the work done per lbm of steam flowing through the turbine. Neglect kinetic and potential energy changes of the steam.



p-v diagram for process of example

Solution

If we construct the *p-v* diagram for the expanding steam, we have the curve shown in the graph. ——— The work done per unit mass is the shaded area of the figure, or $wk_{os} = -\int v\,dp$. This area is a rectangle and a triangle, i.e.

$$
\begin{aligned}
wk_{os} &= (200 - 15)\ \text{lbf/in}^2 \times 144\ \text{in}^2/\text{ft}^2 \times 4\ \text{ft}^3/\text{lbm} \\
&\quad + (200 - 15)(144)\ \text{lbf-ft} \times \frac{1}{2} \times (29.98 - 4.00)\ \text{ft}^3/\text{lbm} \\
&= 106{,}560\ \text{ft-lbf/lbm} + 346{,}054\ \text{ft-lbf/lbm}
\end{aligned}
$$

or

$$wk_{os} \quad 452{,}614\ \text{ft-lbf lbm}$$

or

$$wk_{os} = 581.8\ \text{Btu lbm}$$ *Answer*

We may also calculate this answer by using equation $wk_{os} = - vdp$

We have from the given pressure-volume relation that

$$v = (p - 228.48)\frac{\text{lbm}^2\ \text{ft}^3}{7.12\ \text{lb in}^2 \times \text{lbm}}$$

If we substitute this into equation ——— we get

$$
\begin{aligned}
wk_{os} &= \int_{p_1}^{p_2} (p - 228.48)\frac{1}{7.12}\,dp \\
&= \frac{1}{7.12}\left(\frac{1}{2}p^2 - 228.48p\right)_{p_1}^{p_2} \\
&= \frac{1}{7.12}\left(\frac{1}{2}(15^2 - 200^2) - 228.48(15 - 200)\right) \\
&= \frac{1}{7.12}\left(-19887.5 + 42268.8\right) \\
&= \frac{144\ \text{in}^2\ \text{ft}^2}{7.12\ \text{lbf lbm in}^2\ \text{ft}^3}(22381.3\ \text{lbf}^2\ \text{in}^4) \\
&= 452{,}656\ \text{ft-lbf lbm} = 581.8\ \text{Btu lbm}
\end{aligned}
$$ *Answer*

This answer agrees with the geometric one within an accuracy of 1%.

Example Determine the change in enthalpy per lbm for carbon monoxide (CO) as the gas is cooled from 1500°F to 500°F, assuming the gas does not have a constant specific heat.

Solution We will assume the gas, CO, is a perfect gas so that

$$\Delta h = \int c_p dT$$

and we must find the relation c_p has to temperature.
can see that there is a choice of relations for CO, namely

$$c_p = \left(9.46 - \frac{3.29(10^3)}{T} + \frac{1.07(10^6)}{T^2}\right) \text{ Btu/lbm mole°R}$$

or

$$c_p = [a + b(10^{-3})T + c(10^{-6})T^2 + d(10^{-9})T^3] \text{ cal/g-mole°K}$$

where a, b, c, and d can take on two different sets of values. Let us use the second equation with the following set of constants:

$$a = 6.480$$
$$b = 1.566$$
$$c = -0.2387$$
$$d = 0$$

Then

$$c_p = (6.48 + 1.566\, T \times 10^{-3} - 0.2387\, T^2 \times 10^{-6}) \text{ cal/g-mole°K}$$

and the initial and final temperatures are

$$\begin{aligned} T_1 &= 1500°\text{F} + 460° = 1960°\text{R} \\ &= \frac{5}{9} \times 1960°\text{R} = 1089°\text{K} \end{aligned}$$

and

$$\begin{aligned} T_2 &= 500°\text{F} + 460 = 960°\text{R} \\ &= \frac{5}{9} \times 960 = 533°\text{K} \end{aligned}$$

respectively. We now integrate equation

$$\begin{aligned} \Delta h &= \int_{1089°\text{K}}^{533°\text{K}} (6.48 + 1.566\, T \times 10^{-3} - 0.2387\, T^2 \times 10^{-6})dT \\ &= \left(6.48\, T + 0.783\, T^2 \times 10^{-3} - 0.0796\, T^3 \times 10^{-6}\right)_{1089°\text{K}}^{533°\text{K}} \\ &= [6.48(533 - 1089) + 0.783 \times 10^{-3}(533^2 - 1089^2) - 0.0796 \\ &\quad \times 10^{-6}(533^3 - 1089^3)] \text{ cal/g-mole} \\ &= [-3603 - 706 + 91] \text{ cal/g-mole} \\ &= -4218 \text{ cal/g-mole} \end{aligned}$$

Per gram, the change in enthalpy is found by using the molecular weight of CO, 28 g/g-mole,

$$\begin{aligned} \Delta h &= -4218 \text{ cal/g-mole} \times \frac{1}{28} \text{ g/g-mole CO} \\ &= -150.6 \text{ cal/g} \end{aligned}$$

Answer

This answer could now easily be converted to Btu/lbm if it is desired.

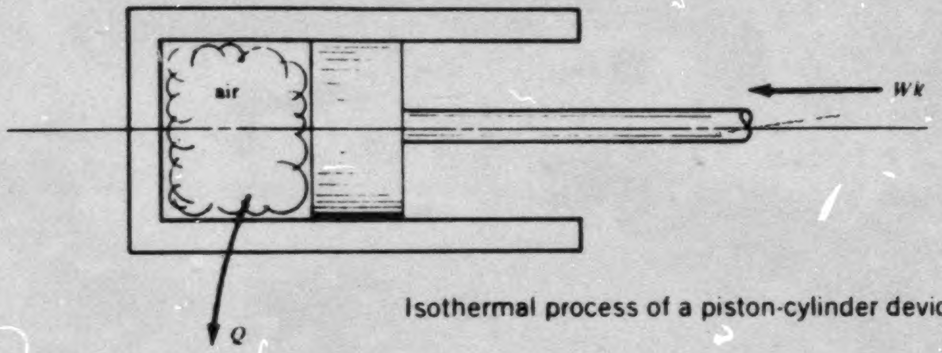
Example

During the compression of 0.01 lbm of air in a cylinder (see figure 6–5), heat is transferred through the cylinder walls to keep the air at a constant temperature. The air pressure increases from 15 psia to 150 psia after the air is fully compressed. The initial specific volume of the air is 7.4 ft³/lbm. Determine the operating air temperature, the change in internal energy and in enthalpy, the work done, and the heat transferred during this process.

Solution

This is an isothermal process and we will assume it to be reversible as well. If the air is behaving as a perfect gas, which we assume, then the operating temperature can be found from

$$T = \frac{pV}{mR} = \frac{pv}{R}$$



Isothermal process of a piston-cylinder device.

or, initially

$$T_1 = \frac{p_1 v_1}{R} = \frac{(15\ \text{lbf in}^2)(7.4\ \text{ft}^3\ \text{lbm})(144\ \text{in}^2/\text{ft}^2)}{53.3\ \text{ft-lbf lbm°R}}$$

so that

$$T_1 = 300°\text{R}$$

and then

$$T_2 = 300°\text{R}$$

The change in internal energy is

$$\Delta U = mc_v\Delta T = 0$$

and for the enthalpy change we have

$$\Delta H = mc_p\Delta T = 0$$

The work done is reversible and this we obtain from equation (6–17),

$$Wk_{cs} = C\ \ln\left(\frac{V_2}{V_1}\right)$$

or, more conveniently, from equation (6–18)

$$Wk_{cs} = C\ \ln\left(\frac{p_1}{p_2}\right)$$

The constant is determined first:

$$\begin{aligned} C &= p_1V_1 = p_1mv_1 \\ &= (15\ \text{lbf in}^2)(0.01\ \text{lbm})(7.4\ \text{ft}^2\ \text{lbm})(144\ \text{in}^2\ \text{ft}^2) \\ &= 159.8\ \text{ft-lbf} \end{aligned}$$

Then

$$\begin{aligned} Wk_{ct} &= (159.8 \text{ ft-lbf}) \ln \frac{15}{150} \\ &= (159.8)\left(-\ln \frac{150}{15}\right) \\ &= 159.8\,(-\ln 10) \\ &= -368 \text{ ft-lbf} \end{aligned}$$ *Answer*

The heat transferred is equal to the work done so that

$$Q = -368 \text{ ft-lbf}$$ *Answer*

and Q is, as the sign indicates, removed from the system. For the irreversible isothermal process, the internal energy change can still be zero; but the work and heat increase in absolute values; that is, more work is required and more heat transfer is demanded to retain constant temperature.

Example 7.5 Air expands polytropically through a nozzle such that the exponent n is to be 1.45. The exhaust pressure of the air is 15 psia and the temperature is 200°F. If the inlet pressure is 60 psig, determine the change in specific entropy of the air as it passes through the nozzle.

Solution For any polytropic process we can use equation

$$\Delta s = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

$$c_p = 0.24 \text{ Btu/lbm°R}$$

and

$$R = 53.3 \text{ ft-lbf/lbm°R} = 0.0686 \text{ Btu/lbm°R}$$

Also, the pressure ratio p_2/p_1 is 15 psia/(60 + 15) psia or $\frac{1}{5}$, assuming an atmospheric pressure of 15 psia. Then,

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{n-1/n}$$

This gives us

$$\frac{T_2}{T_1} = \left(\frac{1}{5}\right)^{0.45/1.45} = \left(\frac{1}{5}\right)^{0.3} = \frac{1}{1.647}$$

Consequently the specific entropy change can easily be determined:

$$\begin{aligned} \Delta s &= (0.24 \text{ Btu/lbm°R}) \ln \frac{1}{1.647} - (0.0686 \text{ Btu/lbm°R}) \ln \frac{1}{5} \\ &= -0.120 \text{ Btu/lbm°R} + 0.111 \text{ Btu/lbm°R} \\ &= -0.009 \text{ Btu/lbm°R} \end{aligned}$$ *Answer*

The entropy is decreasing in this process, due to a drop in the air temperature.

Machine Design

The knowledge gained in applied mechanics on external forces acting on machine elements, the stress calculations from strength of materials, the understanding of the nature and mechanical properties of materials (Metallurgy and Materials), and the methods of producing machine parts (Manufacturing Processes) are all brought together in machine design to determine the dimensions of mechanical components acting as parts of a machine. Information from fluid mechanics and thermal considerations are also used in particular topics encompassed by this course in Machine Design.

The ability to apply mathematics to the machine design field is most emphasized by the need for mathematics in the supporting courses as previously demonstrated.

Complicated problems lead to interesting mathematical equations which will be listed as follows:

Design Example of a Compression Spring (Decimal Exponent)

$$S_{ds} = .324\,\frac{168{,}000}{D_w^{\,.166}}\ (1) \quad \text{also} \quad S_s = k\,\frac{8\,F\,D_m}{\pi D_w^{\,3}}\ (2)$$

$$\text{Equating (1) and (2):}\quad .324\,\frac{168{,}000}{D_w^{\,.166}} = \frac{1.3 \times 8 \times 1500 \times 5.5}{3.14\,D_w^{\,3}}$$

$$\frac{54{,}500}{D_w^{\,.166}} = \frac{27{,}400}{D_w^{\,3}}$$

$$27{,}400\,D_w^{\,.166} = 54{,}500\,D_w^{\,3}$$

$$D_w^{\,2.834} = \frac{27{,}400}{54{,}500} = .502$$

$$D_w = .502^{\frac{1}{2.834}} = .784$$

Elastic Analysis of a Bolted Joint
(The following problem indicates the use of scientific notation and decimals)

$$F_t = F_i + \left(\frac{K_b}{K_b + K_c} \right) F_e$$

$$= 15{,}000 + \left(\frac{1.18 \times \cancel{10^7}}{(1.18 + 2.18) \quad \times \cancel{10^7}} \right) 5000$$

$$= 15{,}000 + 1760 = 16{,}760 \text{ psi} = 1.6760 \times 10^4 \text{ psi}$$

The values of K_b and K_c have already been evaluated from

$$K_b = \frac{A_b E}{L} = \frac{.785 \times 30 \times 10^6}{2} = \frac{7.85 \times 10^{-1} \times 3.0 \times 10^7}{2}$$

$$= 11.8 \times 10^6 = 1.18 \times 10^7$$

and $K_c = \dfrac{A_c E}{L} = \dfrac{4.12 \times 1.06 \times 10^7}{2} = 2.18 \times 10^7$

also $S_b = \dfrac{F_t}{A_b} = \dfrac{1.6760 \times 10^4}{7.85 \times 10^{-1}} = 2.13 \times 10^4$ psi

Then $\Sigma_b = \dfrac{2.13 \times 10^4}{3.0 \times 10^7} = .71 \times 10^{-3} = 7.1 \times 10^{-4}$

and $\delta_b = 1.42 \times 10^3$

Now $\Delta\delta_b = .00142 - .00127 = .00015 = 1.5 \times 10^{-4}$

and $\delta_c = (.00069 - .00015) = .00054 = 5.4 \times 10^{-4}$

$\Sigma_c = .00027 = 2.7 \times 10^{-4}$

$S_c = \Sigma_c E = 28 \times 10^{-5} \times 10.6 \times 10^6 = 2.8 \times 10^{-4} \times 1.06 \times 10^7 = 2.970 \times 10^3$ p

Opening load: $F_o = F_i \dfrac{K_b + K_c}{K_c}$

$$= 15{,}000 \; \frac{(1.18 \times 10^7 + 2.18 \times 10^7)}{2.18 \times 10^7}$$

$$= 15{,}000 \; \frac{(1.18 + 2.18)}{2.18} \times \frac{\cancel{10^7}}{\cancel{10^7}}$$

$$= 15{,}000 \; \frac{(3.36)}{2.18}$$

$$= 1.5 \times 10^4 \, (1.54)$$

$$= 2.31 \times 10^4$$

$$= 23{,}100 \text{ lbs}$$

Combined stresses (cubic equations):

$$S_y = \frac{52{,}000}{2} = \frac{2500}{bh} + \frac{2500 \times (2.5 + \tfrac{1}{2}h) \times 6}{bh^2}$$

$$\frac{52{,}000}{2} = \frac{2500h + 37{,}000 + 7500h}{bh^2} \quad \text{but } h = 3b$$

$$\text{then: } 26{,}000 = \frac{30{,}000b + 37{,}000}{9b^3}$$

$$234{,}000b^3 - 30{,}000b = 37{,}000$$

$$b^3 - .128b - .158 = 0$$

$$\text{Try } b = \tfrac{1}{2}: \ .125 - .064 - .158 = -.394$$

$$b = \tfrac{3}{4}: \ .412 - .096 - .158 \neq +.168$$

by interpolation

$$b = \frac{5}{8}: \ .246 - .079 - .158 \approx 0 - \text{OK}$$

then $h = 3 \times .625 = 1.875$ Ans.

Fractional equations

$$S_m = \frac{\frac{10{,}000}{A} + \frac{D}{A}}{2} = \frac{5{,}000}{A} \qquad A = h \times b = 1.5\,b^2$$

$$S_a = \frac{\frac{10{,}000}{A} - \frac{0}{A}}{2} = \frac{5{,}000}{A}$$

$$\text{then: } \quad \frac{1}{1.4} = \frac{\frac{5{,}000}{A}}{55{,}000} + \frac{\frac{5{,}000}{A}}{36{,}975}$$

$$\frac{A}{1.4} = \frac{5{,}000}{55{,}000} + \frac{5{,}000}{36{,}975}$$

$$\text{solving for A: } \frac{A}{1.4} = .091 + .135 = .226$$

$$A = 1.4 \times .226 = .316$$

Principles of numerically tape- and computer-controlled (N/C) machine tools, in particular the M. T. Department's Burgmaster machining center, are the subject matter of this course. In programming the tape to command the machine tool, dimensions have to be specified to fit the operation of the machine.

The conversion of angular to rectangular dimensions requires a basic knowledge of right-angle trigonometry as illustrated in the example below.

POINT-TO-POINT PROGRAMMING

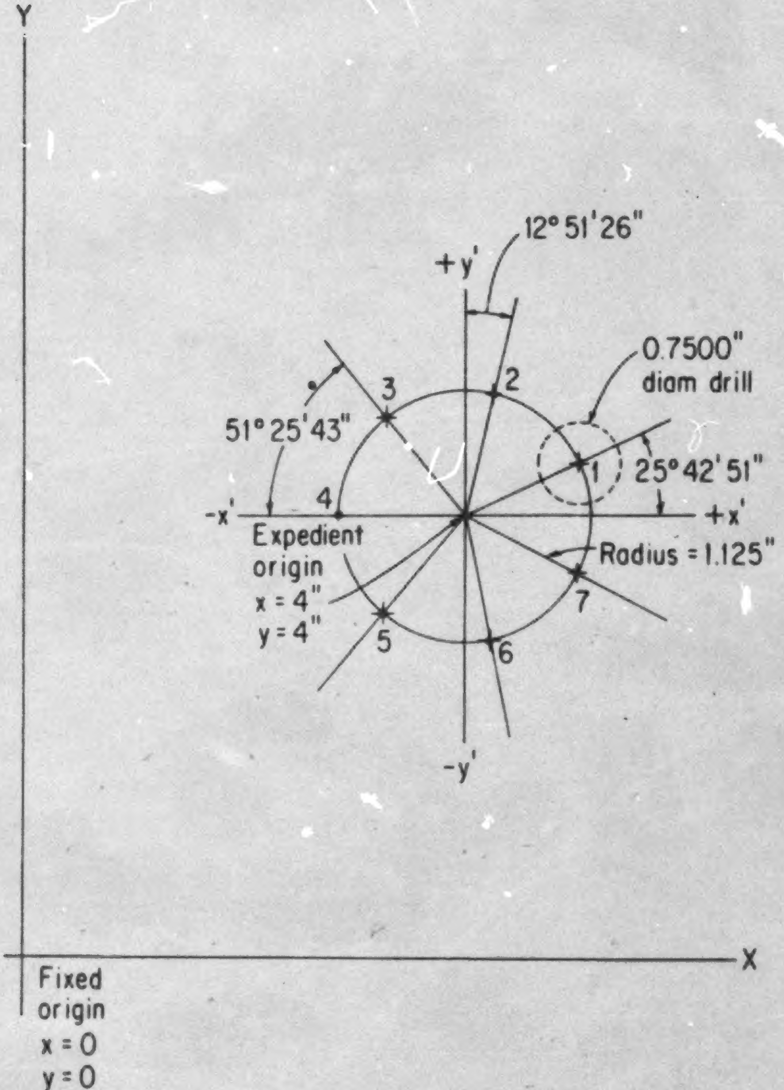

Y
12° 51'26"
+y'
0.7500"
diam drill
2
3
51° 25'43"
1
25° 42'51"
4
-x'
+x'
Expedient
origin
x = 4"
y = 4"
Radius = 1.125"
7
5
6
-y'
X
Fixed
origin
x = 0
y = 0

Fig. 3-10. The x and y coordinate dimensions of points 1 through 7 are calculated by first referring to a set of X' and Y' axes that have been established as an expedient. Assuming that the noted "fixed origin" refers to a fixed point on the machine table, it would be necessary that the center of the part be located at the "expedient origin" which is 4 inches in the $+x$ direction and 4 inches in the $+y$ direction from the fixed Y and X axes, respectively.

Step #1

Determine the angular positions of the hole centers with respect to the X and Y axes:

$$\frac{360^\circ}{7 \text{ holes}} = 51^\circ\ 25'\ 43''$$ angular distance between holes and the angular distance of holes 3 and 5 from the horizontal.

The angular position of hole No. 1 with respect to the horizontal axis is:

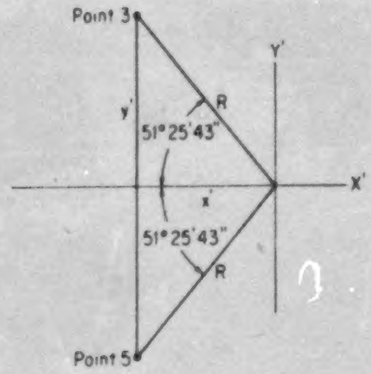
$$180^\circ - 3 \times (51^\circ\ 25'\ 43'') = 25^\circ\ 42'\ 51''$$

The angular position of hole No. 2 with respect to the vertical axis is:

$$90^\circ - 51^\circ\ 25'\ 43'' - 25^\circ\ 42'\ 51'' = 12^\circ\ 51'\ 26''$$

Step #2

Determine the x and y coordinates of the hole centers. This is accomplished by first determining the x' and y' distances from the X' and Y' axes which have been established as an expedient for calculation.



$$y' = R \sin 51^\circ\ 25'\ 43''$$
$$y' = 1.125'' \times .7818 = +.880''$$
$$x' = R \cos 51^\circ\ 25'\ 43''$$
$$x' = 1.125'' \times .6235 = -.701''$$

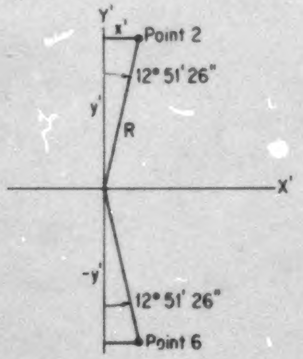
The x' and y' distances for Point #5 would be:

$$x' = -.701''$$
$$y' = -.880''$$

and for Point #3:

$$x' = -.701''$$
$$y' = +.880''$$

Next the x' and y' distances for Points #2 and 6 are found:


Y'
x'
Point 2
12° 51' 26"
y'
R
X'
-y'
12° 51' 26"
Point 6

$$x' = R \sin 12^\circ\, 51'\, 26''$$

$$x' = 1.125 \times .2225 = +.250''$$

$$y' = R \cos 12^\circ\, 51'\, 26''$$

$$y' = 1.125 \times .9749 = +1.097''$$

Point #6 would be at:

$$x' = +.250''$$

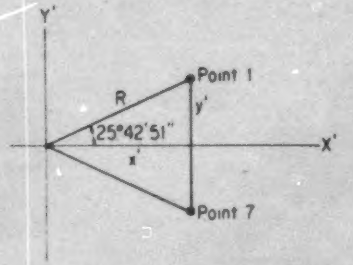
$$y' = -1.097''$$

and Point #2 would be at:

$$x' = +.250''$$

$$y' = +1.097''$$

Next the distances for Points #1 and #7 are found:


Y'
Point 1
R
y'
25°42'51"
X'
x'
Point 7

$$x' = R \cos 25^\circ\, 42'\, 51''$$

$$x' = 1.125 \times .9010 = 1.014''$$

$$y' = R \sin 25^\circ\, 42'\, 51''$$

$$y' = 1.125 \times .4339 = .488''$$

Point #7 would be at:

$$x' = +1.014''$$

$$y' = -.488''$$

and Point #1 would be at:

$$x' = +1.014''$$

$$y' = +.488''$$

Step #3

The distances of the points (hole centers) must next be calculated with respect to the *Fixed Origin* which is 4 inches in the x and y directions from the center of the hole pattern which has been selected as the expedient origin. This is accomplished by either adding 4 inches to the x' and y' dimensions or subtracting the x' and y' dimensions from 4 inches, depending on the point, thus:

Point 1 $x = 4 + 1.014 = 5.014''$

$y = 4 + .488 = 4.488''$

Point 2 $x = 4 + .250 = 4.250''$

$y = 4 + 1.097 = 5.097''$

Point 3 $x = 4 - .701 = 3.299''$

$y = 4 + .880 = 4.880''$

Point 4 $x = 4 - 1.125 = 2.875''$

$y = 4 + 0 = 4.000''$

Point 5 $x = 4 - .701 = 3.299''$

$y = 4 - .880 = 3.120''$

Point 6 $x = 4 + .250 = 4.250''$

$y = 4 - 1.097 = 2.903''$

Point 7 $x = 4 + 1.014 = 5.014''$

$y = 4 - .488 = 3.512''$

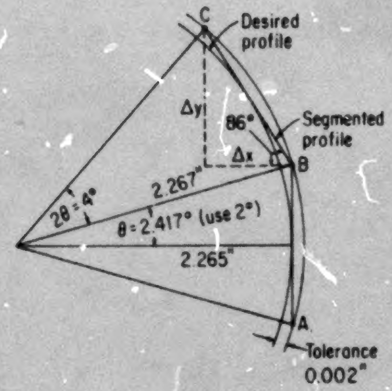


FIG. 4–7. The arc is broken into straight-line segments as shown. Calculation of the segment length is necessary in order to determine the component Δx and Δy movements which must be described on the tape. The lengths of the segments have been exaggerated in order to more clearly demonstrate the calculations.

Referring to Fig. 4–7 which describes an exaggerated segment of the arc:

Mean outside diameter of groove (use) = 4.730″

Mean outside radius of groove = 2.365″

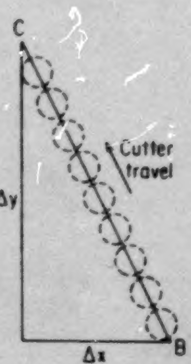
Mean width of groove =
cutter diameter (use) = .200″
cutter radius = .100″

Radius of the cutter path = 2.365″ — cutter radius
= 2.365″ — .100
= 2.265″

Tolerance on radius = $^{+.0035''}_{-.0015''}$

Use $^{+.002}_{-.000}$ for calculations since allowance for machine error must be considered.

Since tape instructions must describe the x and y components of the chord, or straight-line segment, it is first necessary to determine the length of the chord e.g.:



The cutter will travel along the chord line *B–C* although tape instructions denote the distances Δx and Δy, therefore again referring to Fig. 4–7:

$$\text{Cos}\,\theta = \frac{2.265}{2.267} = .99911$$

$$\theta = 2^\circ\,25'$$

$$= 2.417^\circ; \quad \frac{180^\circ}{2.417^\circ} = 74.55 \text{ equal angles}$$

Use 90 angles of 2° each; or 45 angles of 4° each to determine the total length of the straight line segment.

$$\begin{aligned}\text{1/2 the segment length} &= \text{Sin}\,\theta \times 2.267'' \\ &= \text{Sin}\,2^\circ \times 2.267'' \\ &= .0349 \times 2.267'' \\ &= .07912''\end{aligned}$$

$$\begin{aligned}\text{Full segment length} &= 2 \times .07912'' \\ &= .158''\end{aligned}$$

Assuming that the line *A–B* is vertical and therefore parallel to the *Y* axis the tape instructions would be $\Delta x = 0$, since there is no x movement, and $\Delta y = .158''$ which is the length of y travel between *A* and *B* in Fig. 4–7. When calculating the movement of the cutter from *B* to *C* both the Δx and Δy must be calculated.

Δx in this instance would be:

$$\Delta x = .158\ \text{Cos}\,\phi$$

where

$$\phi = 86^\circ$$

Therefore

$$\begin{aligned}\Delta x &= .158\ \text{Cos}\ 86^\circ \\ &= .158 \times .0698 \\ &= .0110''\end{aligned}$$

$$\begin{aligned}\Delta y &= .158\ \text{Sin}\ 86^\circ \\ &= .158 \times .9976 \\ &= .1580''\end{aligned}$$

Design Drafting

Elements of Technology is a remedial course taken concurrently with elementary algebra. The following are illustrations of literal equations encountered.

$F = ma$ Solve for a

$v = \frac{d}{t}$ Solve for d

$P = \frac{Fd}{t}$ Solve for F

$\frac{H}{S} = m(t_a - t_1)$ Solve for m

$I = \frac{E}{R + 2b}$ Solve for E

$D = \frac{CL}{3}$ Solve for L

$C^2 = 3RT$ Solve for R

$A = 2\pi rh$ Solve for r

$P = \frac{W}{(550)(t)}$ Solve for W

$YeA = FL$ Solve for e

$4Dp = \lambda F$ Solve for p

$\frac{N}{n} = \frac{V}{v}$ Solve for V

$LT = \frac{C\theta \pi D^4}{g}$ Solve for θ

$r\lambda E = 377ih$ Solve for i

$F = P(1+rt)$ Solve for P

$I = \frac{nL(PA + pa)}{33{,}000}$ Solve for L

$\frac{mP}{rV} = \frac{C}{S}$ Solve for r

$U = \frac{kMV}{x}$ Solve for x

$\ell = \frac{3\mu}{\rho C}$ Solve for C

$a = \frac{AN^2}{n}$ Solve for n

$C\lambda^3 Z^3 = \frac{\mu}{\rho}$ Solve for ρ

$E = \frac{n^2 h^2}{\int \pi^2 I}$ Solve for I

$(1 - t)(1 + W) = \frac{RV}{(T)(V - v)}$ Solve for T

$S = \frac{WV}{M(100 - t)}$ Solve for M

$\frac{V}{f} = \frac{d}{S}$ Solve for f

$R = \frac{2\gamma}{P}$ Solve for P

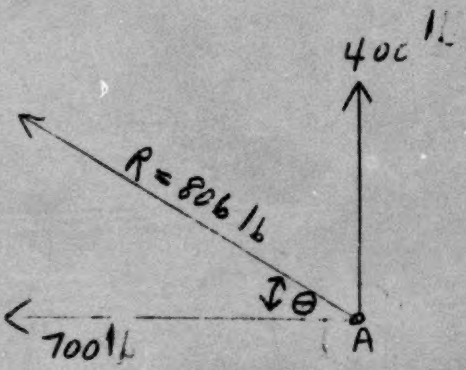
Statics and Strength of Materials

The statics part of the Statics and Strength of Materials course provides a basic understanding of balanced force systems applied to structures and linkages in order to determine the magnitudes of the forces acting on the individual members and supports of a structure and linkage.

The "strength of materials" part of the course analyzes the individual members as to their internal resistance to deformation under the influence of the externally applied forces.

This course for design drafting students, represents a less intense study of a combined version of applied mechanics and strength of materials that is given in the Mechanical Technology curriculum.

1. By calculation, determine the MAGNITUDE and DIRECTION of the RESULTANT of two concurrent forces of 700 lb and 400 lb acting on a body at an angle of 90° with each other as shown. Draw a sketch of the resultant.



Basic equations are:

$x = R \cos \theta \qquad R = (x^2 + y^2)^{1/2}$

$y = R \sin \theta \qquad \theta = \tan^{-1} \frac{y}{x}$

$$R = (400^2 + 700^2)^{1/2} = (16 \times 10^4 + 49 \times 10^4)^{1/2}$$

$$= (65 \times 10^4)^{1/2}$$

$R = 806$ lb

$$\tan \theta = \frac{400}{700} = 0.572$$

$$\theta = 29.8^{\circ}$$

This problem can also be solved graphically by drawing the vectors carefully and measuring R and θ.

2. Find the MAGNITUDE and DIRECTION of the RESULTANT of the concurrent forces shown. Show a sketch of the resultant.

$F_x = F \cos \theta$

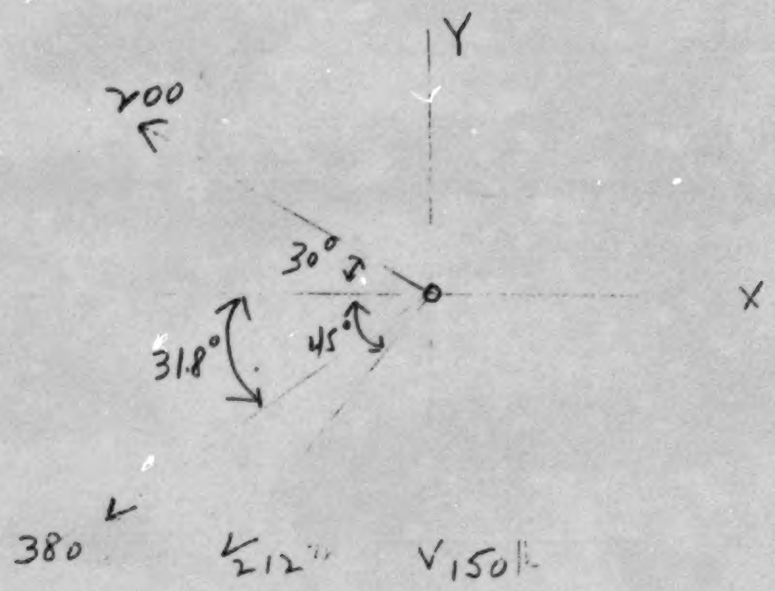
$F_y = F \sin \theta$

$\sin 30^\circ = .500$

$\cos 30^\circ = .866$

$\sin 45^\circ = .707$

$\cos 45^\circ = .707$



Force	θ	Fx	Fy
150 lb	90°	0 lb	−150 lb
200 lb	30°	−173	100 lb
212 lb	45°	−150	−150 lb

$\Sigma F_x = -32.3$ lb $\Sigma F_y = -200$ lb

$R = (323^2 + 200^2)^{1/2}$

$= (10.4 \times 10^4 + 4 \times 10^4)^{1/2}$

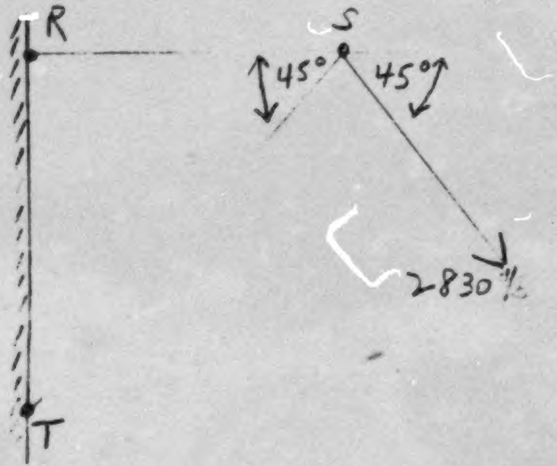
$= (14.4 \times 10^4)^{1/2}$

$R = 380$ lb

$$\tan\theta = \frac{\Sigma F_y}{\Sigma F_x} = \frac{200}{323} = .62$$

$\theta = 31.8^\circ$

3. A load of 2830 lb, making an angle of 45° with the horizontal, is carried by the structure as shown. Find the forces in RS and ST and indicate if the member is in TENSION or COMPRESSION.



$F_x = F\cos\theta \qquad F_y = F\sin\theta$

Force	θ	F_x	F_y
2830 lb	45°	2000 lb	−2000 lb
F_{RS}	0	$-F_{RS}$	0
F_{ST}	45	$-.707F_{ST}$	$-.707F_{ST}$

For equilibrium, the sum of forces in any direction must be zero

$$\therefore\ \Sigma F_x = 0 = 2000 - F_{RS} - .707F_{ST}$$

$$\Sigma F_y = 0 = -2000 - .707F_{ST}$$

$$F_{ST} = \frac{-2000}{.707} = -2830 \text{ lb.}$$

(The direction is opposite the assumed direction.)

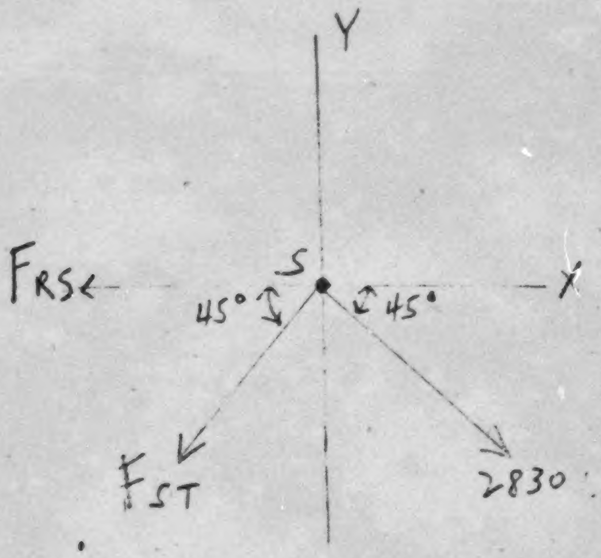
$F_{RS} = 2000 - .707 \times (-2830)$

$F_{RS} = 2000 + 2000 = +4000$ lb

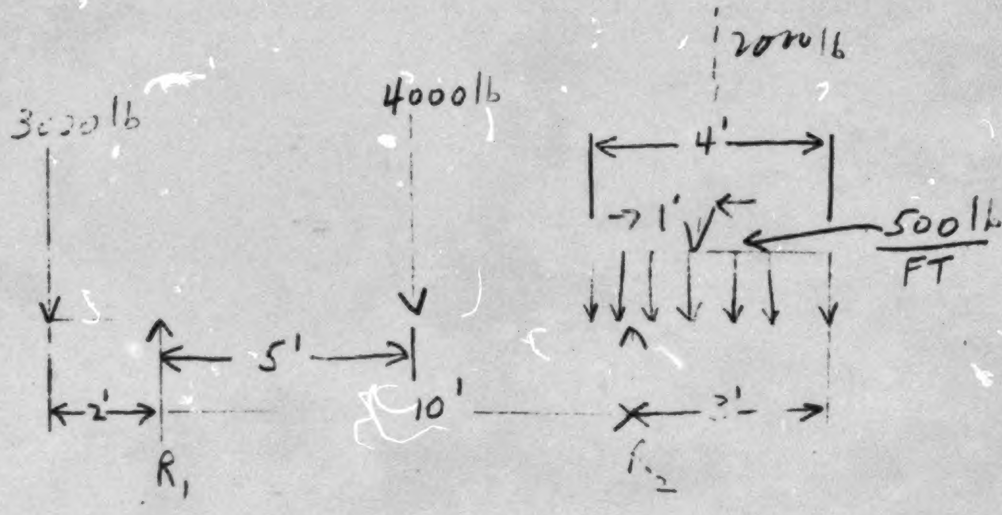
$\therefore F_{ST} = 2830$ lb Compression

$F_{RS} = 4000$ lb Tension

Free body of joint S



4. Determine the MAGNITUDE and DIRECTION of the reactions, R_1 and R_2, for the beam loaded as shown.



The distributed load will weigh 2000 lbs (i.e., 500 lbs x 4 ft) and can be considered to be concentrated at the mid-point, 2 ft from the right side (1 ft from R_2).

$$\Sigma F = 0 = R_1 + R_2 - 3000 - 4000 - 2000$$

$$R_1 + R_2 = 3000 + 4000 + 2000$$

$$R_1 + R_2 = 9000 \text{ lb}$$

$$\Sigma M_{R_2} = 0 = 3000 \times 12 + 4000 \times 5 - 2000 \times 1 - R_1 \times 10$$

A moment is a force times the distance to the point considered.

$$R_1 = \frac{3000 \times 12}{10} + \frac{4000 \times 5}{10} - \frac{2000 \times 1}{10}$$

$$R_1 = 3600 + 2000 - 200 = 5400 \text{ lb}$$

$$R_2 + R_1 - 2000 - 4000 - 3000 = 0$$

$$R_2 = 9000 - R_1 = 9000 - 5400 = 3600 \text{ lb}$$

$\therefore$ $R_1 = 5400$ lb UP, $R_2 = 3600$ lb UP

5. Determine the rod size required to support a 49000 lb tension load if the stress in the rod must not exceed 14000 lb/in^2 and the rods are available only in the diameters listed below.

Diameters Available

½ inch 1 1¼ 1½ 1¾ 2 2¼ 2½ 3

$F = 49000 \text{ lb}$ $\qquad\qquad S_T = 14000 \text{ lb/in}^2$

$$S = \frac{F}{A}$$

$$A = \frac{F}{S} = \frac{49000 \text{ lb}}{14000 \text{ lb/in}^2} = 3.5 \text{ in}^2$$

$$A = \frac{\pi D^2}{4} = 3.5 \text{ in}^2$$

$$D = \sqrt{\frac{4 \times 3.5 \text{ in}^2}{\pi}} = \sqrt{4.46 \text{ in}^2}$$

$$D = 2.11 \text{ in}$$

Use a 2¼ inch diameter rod, the smallest size $\geqslant$ 2.11 in.

6. A 2-inch diameter, aluminum rod, 20 feet long, is going to be used to support a tension load. The toal elongation of the rod must not exceed 0.30 inch, and the stress in the rod must not exceed 15,000 psi. What is the MAXIMUM LOAD that the rod can support so that each of these conditions is satisfied?

$D = 2 \text{ inches}$ $\qquad\qquad E_{\text{aluminum}} = 10.4 \times 10^6 \text{ psi}$

$$A = \frac{\pi D^2}{4} = \frac{\pi (2)^2}{4} = \pi \text{in}^2 = 3.14 \text{in}^2 \qquad \ell = 20 \text{ ft} \times 12 \frac{\text{in}}{\text{ft}} = 240 \text{ inches}$$

$\delta_{\max} = 0.30 \text{ inch}$ $\qquad\qquad S_{\max} = 15{,}000 \text{ psi}$

STRESS

$$S = \frac{F}{A}$$

$$F = S \times A = 15{,}000 \times \pi$$

$$F = 47{,}000 \text{ lb}$$

ELONGATION

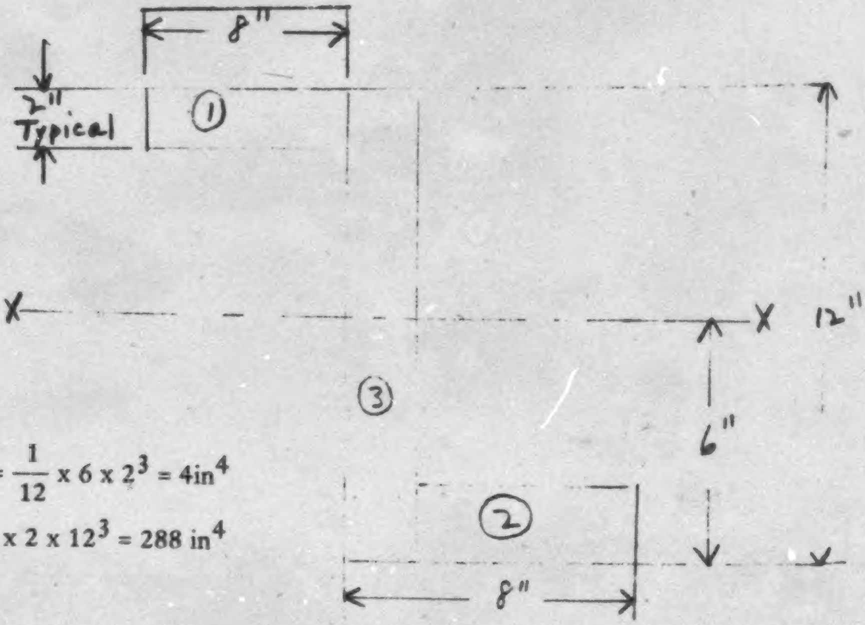
$$E = \frac{F\ell}{A\delta}$$

$$F = \frac{EA\delta}{\ell} = \frac{10.4 \times 10^6 \times \pi \times .30}{240}$$

$$F = \frac{1.04 \times \pi \times 3}{2.4} \times 10^4 = 41{,}000 \text{ lb}$$

$\therefore$ MAXIMUM LOAD = 41,000 lb

7. Find the moment of inertia, I_x, about the horizontal centroidal axis of the section shown. This requires use of the transfer theorem; i.e., $I_z = I_{cg} + md^2$, where I_z is the desired moment of inertia, F_{cg} is the moment of inertia about the center of gravity, m is the mass, and d is the distance from center of gravity to the desired point.



$$I_1 = I_2 = \frac{1}{12} \times 6 \times 2^3 = 4 \text{in}^4$$

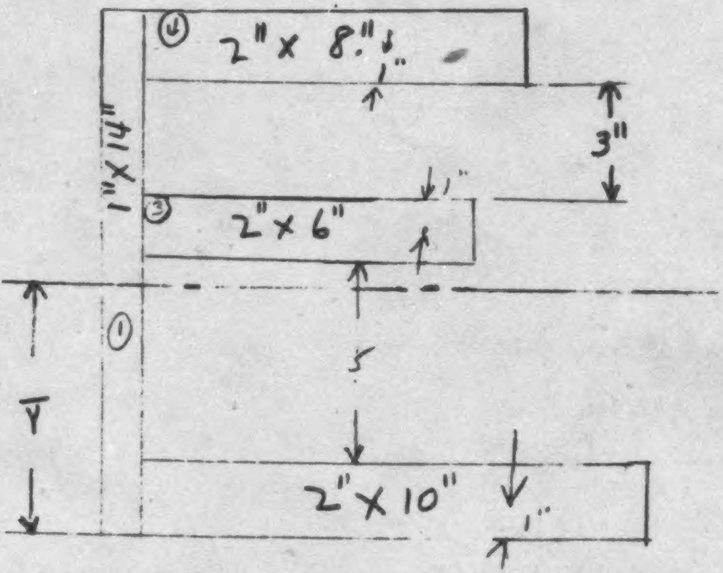
$$I_3 = \frac{1}{12} \times 2 \times 12^3 = 288 \text{ in}^4$$

Area	Dimen.	A	I	d	d^2	Ad^2
1	6 x 2	12	4	5	25	300
2	6 x 2	12	4	5	25	300
3	12 x 2	24	288	0	0	0
			$\Sigma_2 I = 296$			$\Sigma Ad^2 = 600$

$$I_x = \Sigma I + \Sigma Ad^2 = 296 + 600$$

$$\boxed{I_x = 896 \text{ in}^4}$$

For the section made of 4 planks as shown, determine the distance of the centroid above the base. Use dimensions on sketch. The basic assumption is that the entire mass can be considered to be acting at the geometric centroid of the rectangle.



Area	Dimen.	A	Y	A x Y
1	1 x 14	14	7	98
2	2 x 10	20	1	20
3	2 x 6	12	8	96
4	2 x 8	16	13	208

$\Sigma A = 62$ $\Sigma Ay = 422$

$$\bar{y} = \frac{\Sigma Ay}{\Sigma A} = \frac{422}{62} \qquad \boxed{\bar{y} = 6.80 \text{ inches}}$$

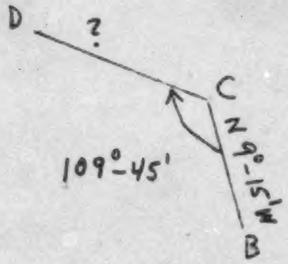
Surveying and Layout

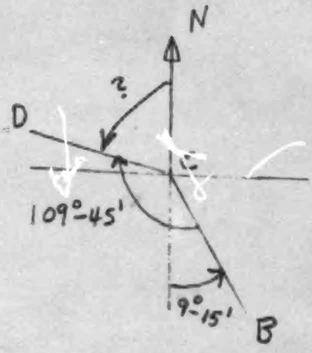
The material in this course consists of an introduction to the fundamental aspects of practical surveying, including both indoor (office) and outdoor (field) work.

The field work consists of the measurement of (1) horizontal distances, using a steel tape and related equipment; (2) angles, using an engineer's transit; and (3) vertical distances (elevations), using an engineer's level and rod.

The office work consists of the computation and mapping of closed traverses, and requires mathematical skills as employed below.

Bearings





180° − 00'
+ 9 − 15

189° − 15'
188° − 75'
− 109° − 45'

$\boxed{\text{N } 79° - 30' \text{ W}}$

Check + Adjustment of Interior Angles

Σ Interior angles = (N − 2) x 180° For a 5-sided polygon, N = 5

Σ Interior angles = (5 − 2) x 180° = $\boxed{540°}$

Allowable error = Least count of vernier x $\sqrt{N}$

For a 1 minute vernier, least count = 1'

For a 5-sided polygon, N = 5

Allowable error = 1' x $\sqrt{N}$ = 1' x $\sqrt{5}$ = $\boxed{2.24 \text{ minutes}}$

Latitudes and Departures

Lat = Length x cosine (bearing) Dep = Length x sine (bearing)

If length = 426.05 ft., bearing = N 79° − 30' W

Lat = + if north
− if south

Dep = + if east
− if west

Lat = 426.05 x cos (79° − 30')
= 426.05 x .18224 = $\boxed{+77.64 \text{ ft}}$
North

Dep = 426.05 x sin (79° − 30')
= 426.05 x .98325 = $\boxed{-418.91 \text{ ft}}$
West

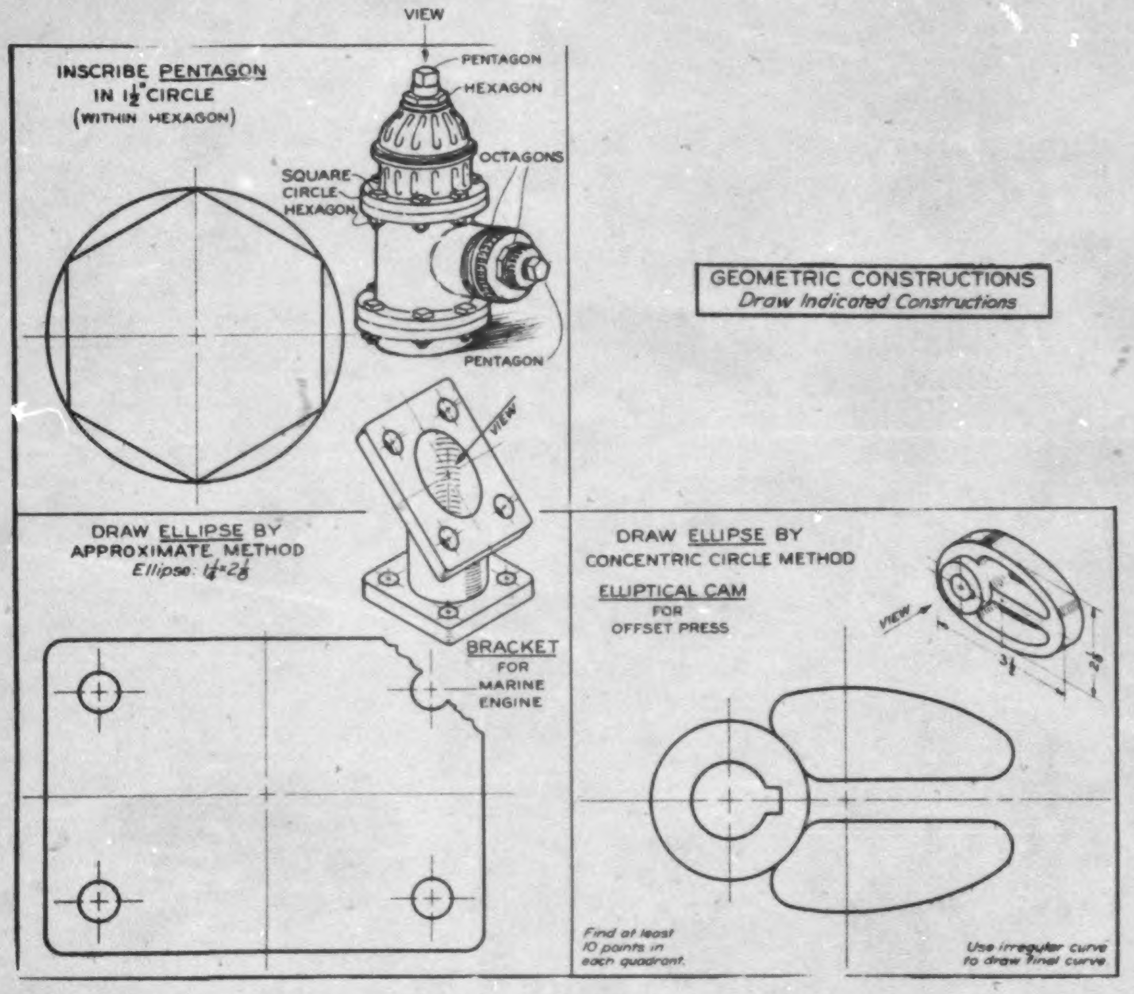
Error of closure = $\sqrt{\Sigma \text{ Lat}^2 + \Sigma \text{ Dep}^2}$ If: Σ Lat = − 0.25 ft Σ Dep = − 0.13 ft

$= \sqrt{(.25)^2 + (.13)^2} = \sqrt{.0794} = \boxed{0.28 \text{ ft}}$

$$\text{Precision} = \frac{1}{\left(\dfrac{\text{Length of survey}}{\text{Error of closure}}\right)} = \frac{1}{\dfrac{3749.46}{.28}} = \frac{1}{13391}$$

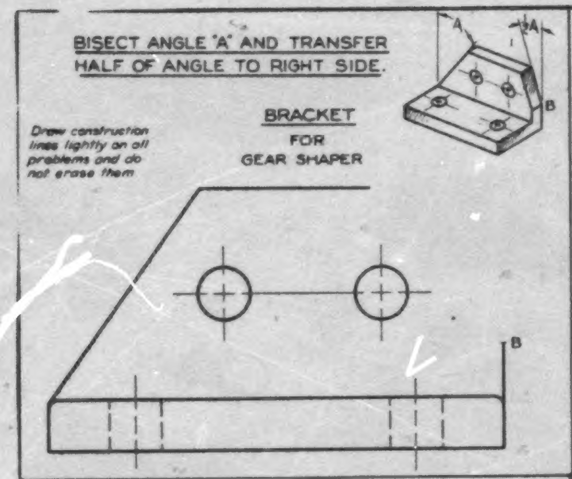
Technical Drawing

The following are typical plates which the drafting student must complete. Can you do them?


VIEW
INSCRIBE PENTAGON
IN 1½" CIRCLE
(WITHIN HEXAGON)
PENTAGON
HEXAGON
OCTAGONS
SQUARE
CIRCLE
HEXAGON
PENTAGON
VIEW
GEOMETRIC CONSTRUCTIONS
Draw Indicated Constructions
DRAW ELLIPSE BY
APPROXIMATE METHOD
Ellipse: 1¼×2⅛
BRACKET
FOR
MARINE
ENGINE
DRAW ELLIPSE BY
CONCENTRIC CIRCLE METHOD
ELLIPTICAL CAM
FOR
OFFSET PRESS
VIEW
1
3¼
2⅛
Find at least
10 points in
each quadrant.
Use irregular curve
to draw final curve.

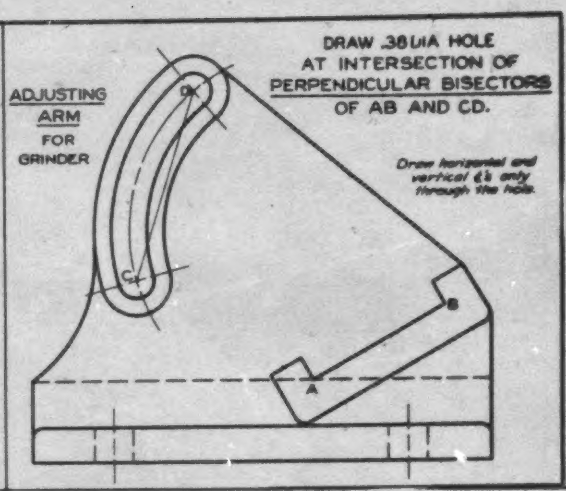
BISECT ANGLE "A" AND TRANSFER HALF OF ANGLE TO RIGHT SIDE.

Draw construction lines lightly on all problems and do not erase them


BRACKET FOR GEAR SHAPER

DRAW .38 DIA HOLE AT INTERSECTION OF PERPENDICULAR BISECTORS OF AB AND CD.

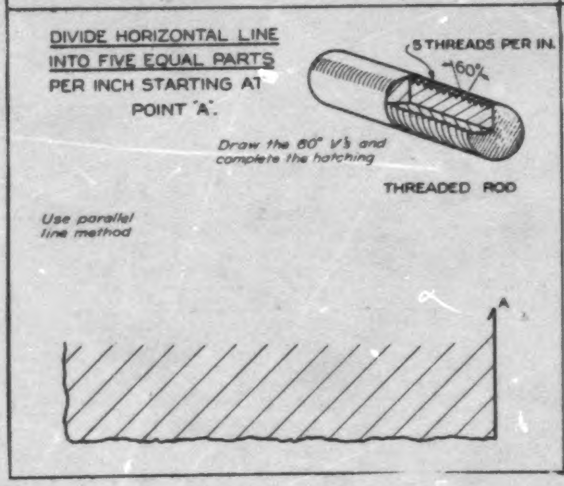
Draw horizontal and vertical ℄'s only through the hole.


ADJUSTING ARM FOR GRINDER

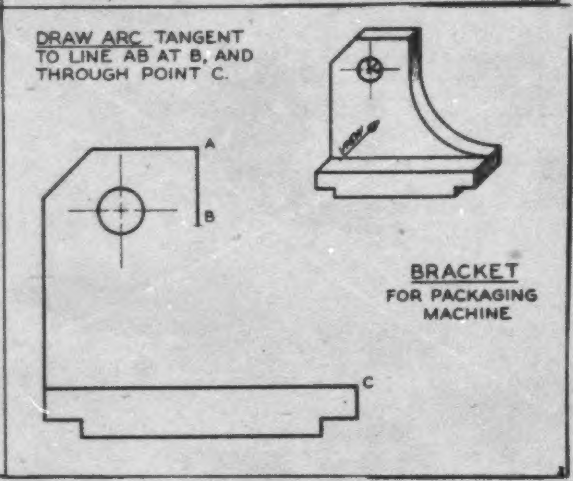
DIVIDE HORIZONTAL LINE INTO FIVE EQUAL PARTS PER INCH STARTING AT POINT "A".

Use parallel line method

Draw the 60° V's and complete the hatching


THREADED ROD

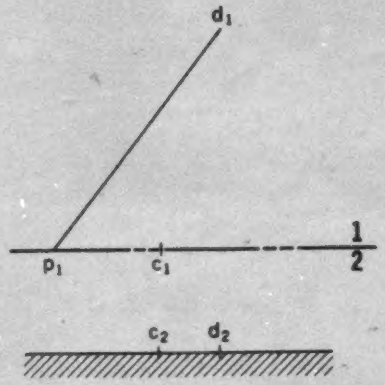
DRAW ARC TANGENT TO LINE AB AT B, AND THROUGH POINT C.


BRACKET FOR PACKAGING MACHINE

SHORTEST DISTANCE FROM A POINT TO A LINE

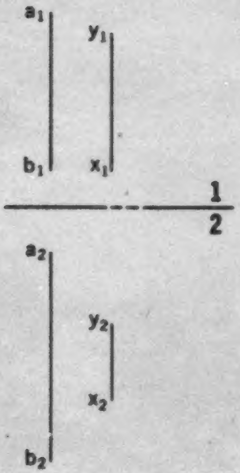
DP is a drain pipe making an angle of 25° with the floor (*P* is above floor level). What is the shortest length of pipe for a second drain connecting the first pipe to point *C*, and where is the point of connection? Scale: ¼ in. = 1 ft-0 in.

Length = ________



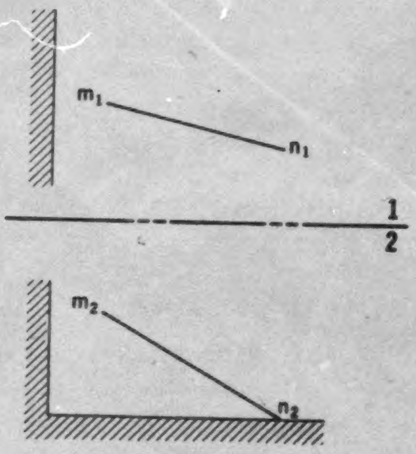
PARALLEL LINES

Are the lines *AB* and *XY* parallel? ________. If not, construct a line *XZ* equal in length to *XY* and parallel to *AB*.



ROTATION

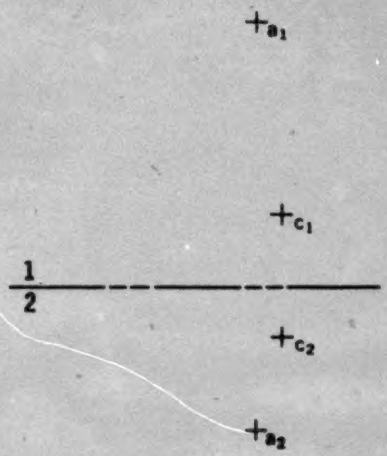
MN represents a control shaft. What is the largest diameter of a handwheel that can be attached perpendicular to the shaft at *M* so that the handwheel will clear the wall and floor when it is rotated? Scale: 1 in. = 1 ft-0 in.



Diameter = ________

TRUE LENGTH, GRADE, AND BEARING OF A LINE

From point *A*, a tunnel bears S45°W on a rising grade of 20% for a distance of 150 ft. A second tunnel, starting from point *C*, is to connect to the first tunnel at a point 50 ft from *A*. For the second tunnel, determine its top and front projections, its length, bearing, and grade. Scale: 1 in. = 100 ft-0 in.



Length = ________
Bearing = ________
Grade = ________

Mechanisms

This course deals with the relative motions and velocities of machine parts and with their accelerations. The application of the principles of motion geometry (kinematics) to the analysis and design of useful mechanisms is mainly graphical in nature, but some use of mathematics is shown in the sample problems given below.

Ball Bearings

At a given load, the rpm is inversely proportional to the life; in other words, doubling the rpm means halving the life, and so on.

Changes in load, on the other hand, have far greater influence on bearing life, since for ball bearings, the life varies inversely as the third power of the load (load^3). For roller bearings, the power is 3.3 (load$^{3.3}$).

This relationship may be expressed in the following formula:

$$\frac{(\text{Rated load})^3}{(\text{Actual load})^3} = \frac{\text{Actual life, revolutions}}{10^6 \text{ (i.e., rated life, rev.)}}$$

The following example will clarify the use of this formula:

A radial bearing has a rated load capacity of 130 lbs for a life of 10^6 rev. What will the life be if the load is increased from 130 to 150 lbs, all other factors remaining the same:

Solution: $$\frac{(130)^3}{(150)^3} = \frac{\text{Act. life, rev.}}{10^6}$$

Act. life ≈ 650,000 rev.

Speed Ratio of a Gear Train (Tooth Ratio — t.r.)

Combining speed ratios into a single overall speed ratio for an entire gear train:

$$\frac{\omega_F}{\omega_E} = \frac{T_E}{T_F}$$

But $\omega_E = \omega_C$, so, substituting ω_C for ω_E,

$$\frac{\omega_F}{\omega_C} = \frac{T_E}{T_F}$$

Since $\frac{\omega_C}{\omega_B} = \frac{T_B}{T_C}$ then: $\omega_C = \omega_B \times \frac{T_B}{T_C}$

Substituting this for ω_C:

$$\frac{\omega_F}{\omega_B \times \frac{T_B}{T_C}} = \frac{T_E}{T_F} \quad \text{or} \quad \frac{\omega_F}{\omega_B} = \frac{T_B}{T_C} \times \frac{T_E}{T_F}$$

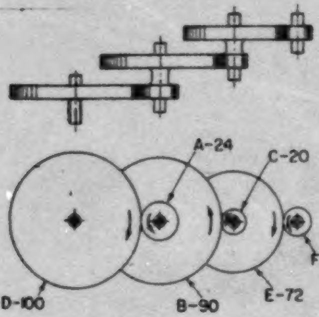
But $\omega_B = \omega_A$, so: $\frac{\omega_F}{\omega_A} = \frac{T_B}{T_C} \times \frac{T_E}{T_F}$ Next, $\frac{\omega_A}{\omega_D} = \frac{T_D}{T_A}$ or $\omega_A = \omega_D \times \frac{T_D}{T_A}$

Substituting again: $\frac{\omega_F}{\omega_D \times \frac{T_D}{T_A}} = \frac{T_B}{T_C} \times \frac{T_E}{T_F}$

And the overall speed ratio: $\frac{\omega_F}{\omega_D} = \frac{T_D \times T_B \times T_E}{T_A \times T_C \times T_F}$

Gears D, B, and E are all drivers, and A, C, and F are all followers of their respective pairs. The right-hand side of the above equation could therefore be described as the product of the tooth numbers of all driver gears divided by the product of the tooth numbers of all follower gears. For simplicity this expression may be called the tooth ratio (symbol t.r.).

TRAINS OF DIFFERENT TYPES OF GEARS



Velocity

Given: The mechanism shown in Fig. 9a at half-scale. In the position shown, crank AB has an instantaneous, counterclockwise angular velocity of 3 radians per second.

Required: Direction and magnitude of the instantaneous linear velocity of point E on link BD.

Solution: First find the magnitude and direction of two velocity vectors on link BD (at B and D) by effective components. $V_B = r = 3 \times 2.63 = 7.89$ inches per second. Let us select a scale such that V_B is represented by a vector of 1.5 inches long. After construction of V_D, we find it has a scale length of 1.93. The actual linear velocity of V_D is then:

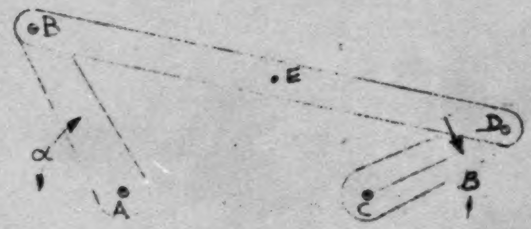
$$\frac{1.93}{1.5} \times 7.89 = 10.15 \text{ inches per second.}$$

By extending cranks AB and CD downwards, we find the instant center of rotation O. Draw OE and erect a perpendicular to it at E in the direction of motion. We can now find the scale value of V_E from:

$$\frac{V_E}{EO} = \frac{V_B}{OB} \quad \text{or} \quad V_E = \frac{EO \times V_B}{OB} = \frac{3.2 \times 1.5}{4.125} = 1.16$$

The actual value of V_E is again found from

$$\frac{1.16 \times 7.89}{1.5} = 5.1 \text{ inches per second (Ans.)}$$



AB = 2.63 CD = 2.25 AC = 3.5 DE = 3.5

$\alpha = 60^{\circ}$ $\beta = 25^{\circ}$

In the design of linkages, it may be necessary to determine the dimensions of a crank, connecting rod, or follower arm, when any two of the three are given. This is often possible by construction. The following example shows how a typical construction may be performed.

Given: Center distance and locations of center of crank r_C and follower crank r_D extreme positions AD and BD of follower crank, and its length.

Required: Length of driver crank r_C and of connecting rod R.

Solution: Draw the two extreme positions of driver crank r_C and connecting rod R, in which they are aligned and in which their centerlines both pass through the center of rotation of r_C. Draw BC first, then AC. BC now equals $R + r_C$, while AC equals $R - r_C$. Since $(R + r_C) - (R - r_C) = 2r_C$, graphically subtracting AC from BC yields $2r_C$. The crank circle can now be drawn and R found as the distance from the intersection E of the crank circle of r_C with BC, to B.

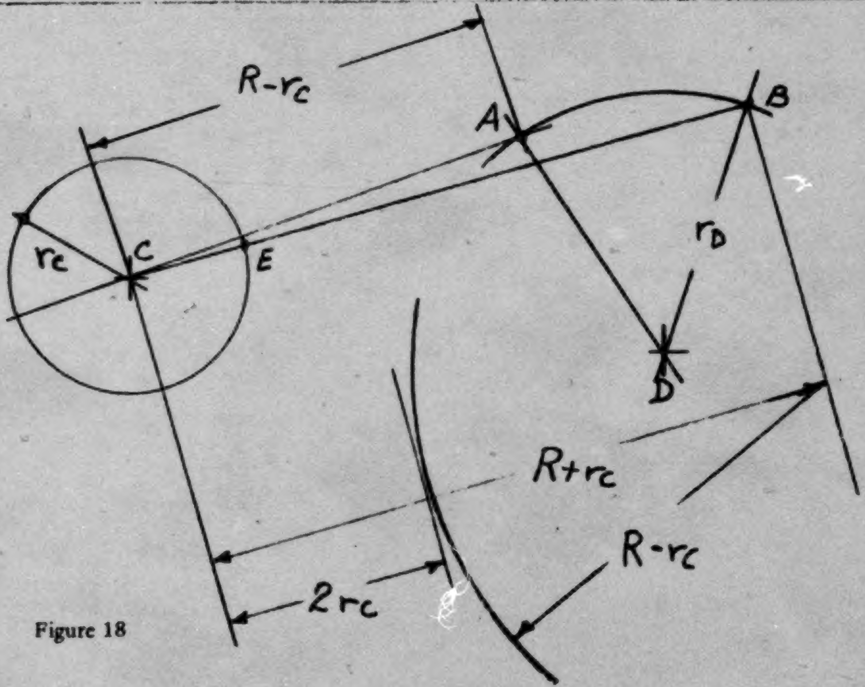
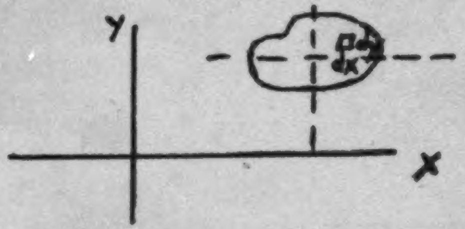


Figure 18

Illustrative Uses of Mathematics Not Previously Covered

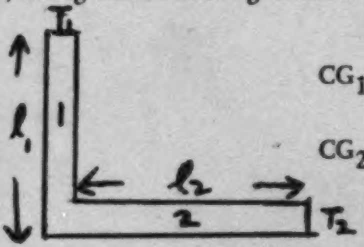
The moment of a force about a point is the force multiplied by the perpendicular distance from the point to the force, or the moment varies jointly as the force and distance. Note that if a door knob were placed in the middle of a door, it would require double the force that a knob placed at the edge of the door would require to obtain the same moment. Levers, claw hammers, pliers, etc., use this principle.

If we had a solid object with distributed weight, we could calculate the moments created by each little piece of the solid, and, by summing them, obtain the moment of the entire object. As usual, we would in the limit go from a sum to an integral. If the density and thickness of the solid are constant, the moment is basically determined by the shape. If the figure is symmetric, we can see that the moments on both sides of a line of symmetry will cancel each other out. Non-symmetric bodies will have at least two lines about which the moments sum to zero. The intersection of these lines is the center of gravity, or, in the case of constant density and thickness, the center of area of the body. Note that the cg may not divide the total area in half, or in some cases even be on the body.



The concept of center of gravity is of crucial importance in analyzing the motion of an object, since the object can be replaced by a point at the cg with all forces acting at that point. If we have a body made up by composition of several parts, we can find the total moment by taking moments of each part, by taking weight or area times distance to the cg and summing. We can also use the moments and the weight or area to find the cg. In the case of constant density and thickness, the cg will be at the geometric center.

Example: Find the cg of an L-Beam



CG_1: $\left(\frac{T_1}{2}, \frac{L_1}{2}\right)$ $A = \ell_1 T_1$

CG_2: $\left(T_1 + \frac{\ell_2}{2}, \frac{T_2}{2}\right)$ $A = \ell_2$

$$M_x = \frac{\ell_1}{2}(\ell_1 T_1) + \frac{T_2}{2}(\ell_2 T_2) = \bar{y}(\ell_1 T_1 + \ell_2 T_2)$$

$$\bar{y} = \frac{\ell_1^2 T_1 + \ell_2 T_2^2}{2(\ell_1 T_1 + \ell_2 T_2)}$$

$$M_y = \frac{T_1}{2}(\ell_1 T_1) + (T_1 + \frac{\ell_2}{2})(\ell_2 T_2) = \bar{x}(\ell_1 T_1 + \ell_2 T_2)$$

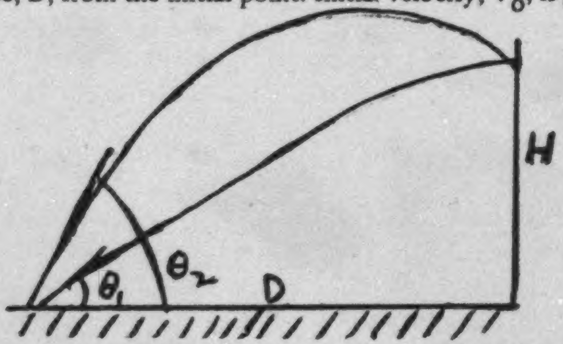
$$\bar{x} = \frac{\ell_1 T_1^2 + 2\ell_2 T_1 T_2 + \ell_2^2 T_2}{2(\ell_1 T_1 T \ell_2 T_2)}$$

For the special case when the beam is symmetric, $t_1 = t_2$ and $\ell_2 = \ell_1 - t_1$

$$\overline{X} = \frac{\ell_1 t_1 + \ell_1^2 - t_1^2}{2(2\ell_1 - t_1)}, \quad \overline{Y} = \frac{\ell_1^2 + \ell_1 t_1 - t_1^2}{2\,(2\ell_1 - t_1)}$$

When $t_1 << \ell_1$, we can approximate $\overline{X}$ and $\overline{Y}$, $\overline{X} = \frac{\ell_1}{4}$, $\overline{Y} = \frac{\ell_1}{4}$

Problem: To find the inital angle, θ, of a trajectory which will hit a target at a specific height, H, and distance, D, from the initial point. Initial velocity, V_o, is given.



The standard equations of motion, neglecting air resistance are:

$$X = V_o \cos\theta\, t; \qquad Y = -\tfrac{1}{2}gt^2 + V_o \sin\theta\, t$$

$$D = V_o \cos\theta\, t \text{ and } t = \frac{D}{V_o \cos\theta}$$

$$H = -\tfrac{1}{2}gt^2 + V_o \sin\theta\, t = -\tfrac{1}{2}\frac{gD^2}{V_o^2} \sec^2\theta + D\tan\theta$$

Since $\sec^2\theta = \tan^2\theta + 1$, we obtain $\frac{gD^2}{2V_o^2} \tan^2\theta - D\tan\theta \, \frac{gD^2}{2V_o^2} + H = 0$

$$\text{Tan}\,\theta = D \pm \frac{\sqrt{D^2 - \frac{2gD^2}{V_o^2}\left(\frac{gD^2}{2V_o^2} + H\right)}}{\frac{gD^2}{V_o^2}}$$

Given: $V_o = 100$ ft/sec, $g = 32.2$ ft/sec^2, $D = 100$ ft.

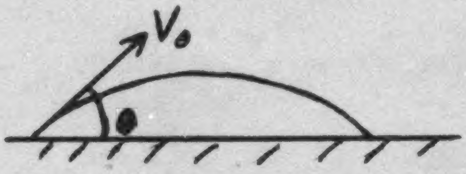
1. $H = 20$ ft, $\theta = 76.6^o$
2. $H = 80$ ft, $\theta = 77.3^o$ or 29.5^o
3. $H = 150$ ft. No solution.

We see that if the target is too high (3), there will be no solution, while under certain conditions (2), there will be two solutions – this means that the projectile strikes the target going up or coming down.

If the discriminant is negative, there will be no solution. If the discriminant is positive, there will be two solutions. If the discriminant is zero, there is one solution.

Problem: Maximum and minimum without calculus.

1. Find the maximum range of a projectile fired with initial velocity V_0.



The range depends upon θ. If θ is 0 or π, the range will be zero.

$X = V_0 \cos\theta\, t$ $\qquad\qquad$ $Y = -\frac{1}{2} gt^2 + V_0 \sin\theta\, t$

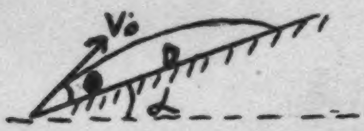
At the end of the flight, $Y = 0$ and $t \neq 0$, or

$$\frac{2V_0 \sin\theta}{g} - t = 0.$$

$$\text{Then } X = \frac{(V_0 \cos\theta)(2V_0 \sin\theta)}{g} = \frac{2V_0^2 \sin\theta\cos\theta}{g} = \frac{V_0^2 \sin 2\theta}{g}$$

This will be a maximum when $\sin 2\theta = 1$ or $\theta = \frac{\pi}{4}$ and $X = \frac{V_0^2}{g}$

If the projectile is fired up a plane inclined α to the horizontal with initial velocity V_0, what is the maximum distance up the plane? Call the distance D.



$X = V_0 \cos\theta\, t$ and $Y = -\frac{1}{2} gt^2 + V_0 \sin\theta\, t$. Also at landing,

$X = D\cos\alpha$, $Y = D\sin\alpha$

$$Y = \frac{-gX^2}{2V_0^2 \cos^2\theta} + X\tan\theta$$

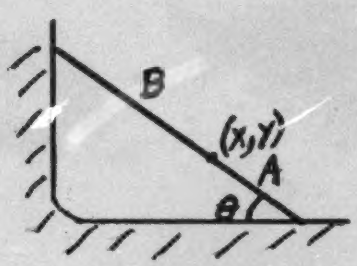
$$\frac{D\sin\alpha}{2V_0^2\cos^2\theta} = \frac{-D^2\sin^2\alpha}{2V_0^2\cos^2\theta} + D\cos\alpha\tan\theta$$

$$D = 2V_0\cos\theta\,\frac{\cos\alpha\sin\theta - \sin\alpha\cos\theta}{\sin^2\alpha} = \frac{2V_0\cos\theta\sin(\theta-\alpha)}{\sin^2\alpha} = \frac{V_0\sin\alpha\sin(2\theta-\alpha)}{\sin^2\alpha}$$

This is the maximum when $\sin(2\theta - \alpha) = 1$ or $2\theta - \alpha = \frac{\pi}{2}$ or $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$

$$\text{and } X = \frac{V_0}{\sin\alpha}$$

Problem: Consider the problem of the path of a point on a ladder, whose base is being pulled out while the top is sliding down a vertical wall. Find the path of a point on the ladder. Let the point be distance A from the base, and B from the top, making the total length A + B.



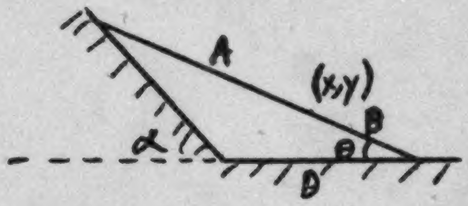
Let θ be the angle between the ladder and the floor. Then

$$Y = A \sin\theta \qquad X = (A + B)\cos\theta - A\cos\theta = B\cos\theta$$

This the equation of an ellipse in Cartesian Coordinates, i.e.,

$$\frac{Y^2}{B^2} + \frac{X^2}{A^2} = 1$$

Problem: Suppose instead of a vertical wall, the wall is slanted at an angle α.



$$X = D - B\cos\theta \qquad Y = B\sin\theta$$

By the Law of Sines, $\dfrac{A+B}{\sin\alpha} = \dfrac{D}{\sin(\theta-\alpha)}$

$$D = \frac{(A+B)\sin(\theta-\alpha)}{\sin\alpha}$$

$$X = \frac{(A+B)\sin(\theta-\alpha)}{\sin\alpha} - B\cos\theta$$

$$X = \frac{(A+B)}{\sin\alpha}(\sin\theta\ \cos\alpha - \sin\alpha\ \cos\theta) - B\cos\theta$$

$$X = \frac{(A+B)}{\sin\alpha}\left(\sqrt{1-\frac{Y^2}{B^2}}\cos\alpha - \frac{Y}{B}\sin\alpha\right) - B\sqrt{1-\frac{Y^2}{B^2}}$$

This is a conic section, whose form will depend upon the values of A, B and α.

Problem: Given a solid figure (1) whose sides are fairly regular and whose top, bottom and mid-section are parallel (they can be points), and whose height is H, the volume is approximately

$$V = \frac{H}{6}(A_b + 4A_m + A_t)$$

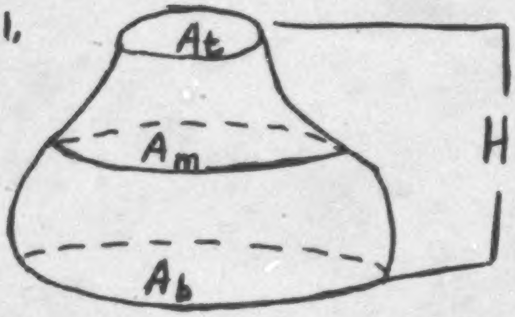
Example: find the volume of the intersection of two equal cylinders (2), whose radii are R.

$A_b = A_t = 0. \quad A_m = (2R)(2R) = 4R^2$

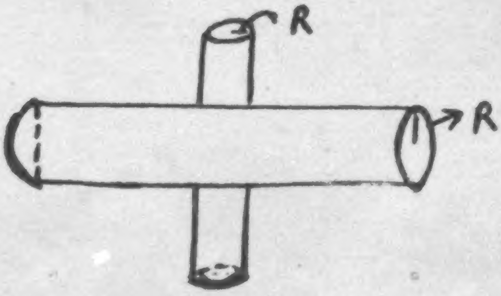
$H = 2R$

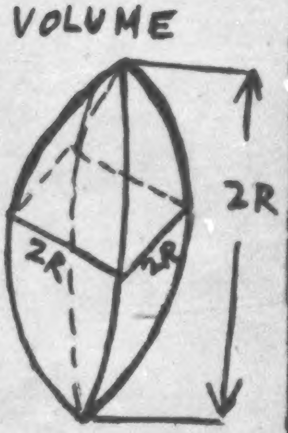
$$V = \frac{2R\,(16R^2)}{6} = \frac{16R^3}{3}$$

1,



2.





Problem: Consider the problem of an object thrown into the air with initial velocity V_o, neglecting air resistance.

The height at any time is: $H = -\frac{1}{2}gt^2 + V_ot$

and the velocity is: $V = -gt + V_o$.

If one wishes to find the velocity at any height, the Height equation is solved for t, and this value is then substituted into the velocity equation.

The time required to reach any height is:

$$t = \frac{V_o}{g} \pm \frac{\sqrt{V_o^2 - 2gH}}{g}$$

Note: If the discriminant is negative, the height will never be reached; i.e., the initial velocity is not sufficient to overcome the pull of gravity.

If the discriminant is positive, it indicates that the projectile will achieve this height twice – once going up and once going down.

If the discriminant is zero, the height will be reached just once; this is the maximum height achieved. (V at this point will be zero.)

The time will be $t = \frac{V_o}{g}$ and the height will be $H = \frac{V_o^2}{g}$.

In general, the relation between height and velocity will be:

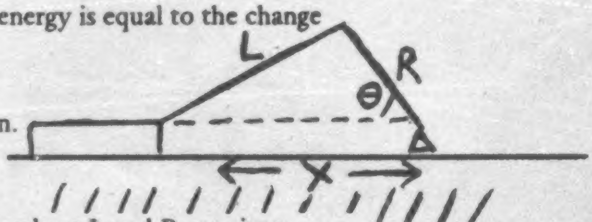
$$2gH = V_o^2 - V^2$$

If we multiply both sides by mass (m) and divide by 2, we obtain:

$$mH = \frac{m(V_o^2 - V^2)}{2}$$

Physically, this means that the change in potential energy is equal to the change in kinetic energy.

Problem: Consider the motion of a slider mechanism as shown.



By the Law of Cosines, $L^2 = X^2 + R^2 - 2RX\cos\theta$, where L and R are given. Differentiating implicitly with respect to time, we obtain:

$$0 = 2X\frac{dX}{dt} - 2R\cos\theta\frac{dX}{dt} + 2RX\sin\theta\frac{d\theta}{dt}$$

$$\frac{dX}{dt} = -\frac{2RX\sin\theta}{2R\cos\theta - 2X}\frac{d\theta}{dt}$$

Since $X = R\cos\theta \pm \sqrt{R^2\cos^2\theta - R^2 + L^2} = R\cos\theta \pm \sqrt{L^2 - R^2\sin^2\theta}$,

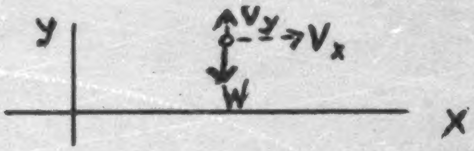
$$\frac{dX}{dt} = \frac{R\sin\theta\,(R\cos\theta \pm \sqrt{L^2 - R^2\sin^2\theta})}{\mp\sqrt{L^2 - R^2\sin^2\theta}}\frac{d\theta}{dt}$$

Note: Even if $\frac{d\theta}{dt}$ = Const., $\frac{dX}{dt}$ will vary with θ

$\frac{dX}{dt} = 0$ when $\theta = 0, \pi, 2\pi$, the extremal points of motion.

Problem: An often overlooked area of mathematics is that of parametric equations. They are useful under the following conditions: (1) The parameter has physical meaning, time slope, angle. (2) The use of a parameter is more convenient than a direct relation between variables. (3) The range of the variables can be limited by the use of a parameter.

The flight of a projectile with only the effect of gravitational forces can best be described by considering the horizontal and vertical motions separately and as functions of time.



The equations of motion based upon F = ma are:

$$O = \frac{W}{g}\frac{d^2x}{dt^2} \quad \text{and} \quad -W = \frac{W}{g}\frac{d^2y}{dt^2}, \text{ leading to}$$

$$X = V_x t + S_x \quad \text{and} \quad Y = \frac{-gt^2}{2} + V_y t + S_y$$

If we eliminate the parameter t, we obtain :

$$Y = \frac{-g\,(X - S_x{}^2)}{2V_x{}^2} + \frac{V_y\,(X - S_x)}{V_x} + S_y\,, \text{ a parabola.}$$

Problem: Consider the accurate point-by-point plotting of an ellipse. In Cartesian Coordinates, this would lead to solving the equation:

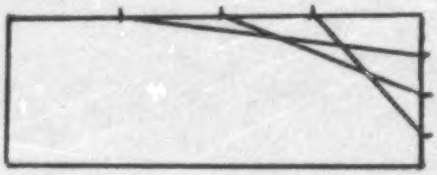
$$Y = \frac{B}{A}\sqrt{A^2 - X^2}\,,$$

which involves a combination of multiplying and subtracting. If we choose parametric equations, we obtain:

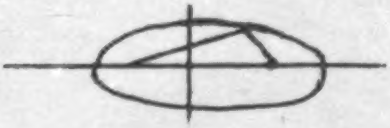
$$X = A\cos\theta,\quad Y = B\sin\theta$$

with A and B the horizontal and vertical semi-axes.

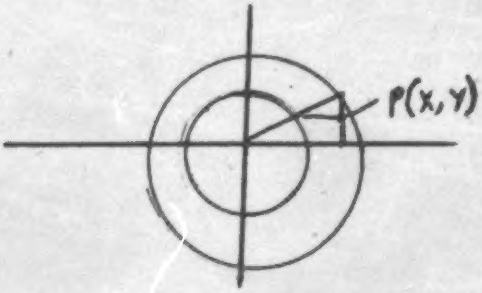
One can also draw a quarter ellipse by drawing a rectangle of sides A and B, and dividing two adjacent sides into the same number of equally divided parts. Connect the points and one obtains a family of tangents to the ellipse.



The ellipse can also be drawn by using the basic definition that it is the locus of points the sum of whose distances from two fixed points is constant. Find the points (foci) at $\sqrt{A^2 - B^2}$, and measure a string at length 2A. Tacking the ends of the string at the foci, hold the string taut with a pencil and swing the ellipse.



Finally, one can draw two concentric circles with radii B and A. At any point on the inner circle, draw a horizontal line and a line from the origin. Where the origin line meets the outer circle, drop a vertical. The intersection of the horizontal and vertical lines is on the ellipse.

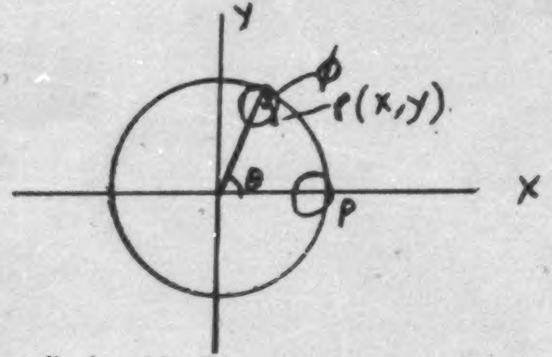


Translation by h & k gives, $x = h + A\cos\theta$ and $y = K + B\sin\theta$.

Note: The hyperbolas would be expressed as:
$X = A\sec\theta$, $Y = B\tan\theta$ or $X = A\tan\theta$, $Y = B\sec\theta$,
depending on their opening horizontally or vertically.

Problem: The Hypocycloid.

Consider the problem of one gear rolling without slipping about the inside of a fixed gear. The large gear has radius A, the small one radius B.



Since there is no slipping, $A\theta = B\phi$

$$X = (A - B)\cos\theta + B\cos(\phi - \theta) = (A - B)\cos\theta + B\cos\frac{(A-B)\theta}{B}$$

$$Y = (A - B)\sin\theta - B\sin(\phi - \theta) = (A - B)\sin\theta - B\sin\frac{(A-B)\theta}{B}$$

If we let $A = 3B$, we obtain:

$X = 2B\cos\theta + B\cos 2\theta$ $\qquad$ $Y = 2B\sin\theta - B\sin 2\theta$

A particularly interesting case arises when $A = 4B$:

$$X = 3B\cos\theta + B\cos 3\theta = 4B\cos^3\theta = A\cos^3\theta$$

$$Y = 3B\sin\theta - B\sin 3\theta = 4B\sin^3\theta = A\sin^3\theta$$

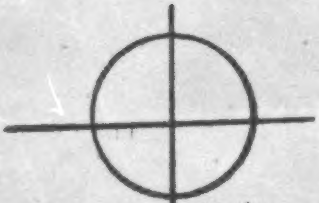
Since $\cos\theta = (\frac{X}{A})^{2/3}$ and $\sin\theta = (\frac{Y}{A})^{2/3}$, and since $\cos^2\theta + \sin^2\theta = 1$, we obtain:

$$X^{2/3} + Y^{2/3} = A^{2/3}$$

Problem: Polar Coordinates are useful for circles fitting the following conditions:

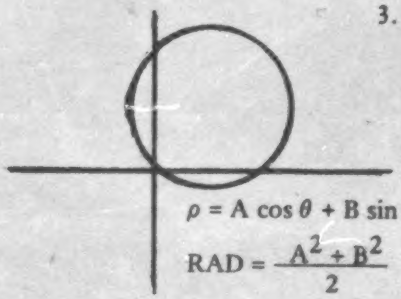
(1) Center at the origin, radius A; (2) Through the origin; (3) Through the origin with radius A and center at (A/2,O); (4) Through the origin with radius A and center at (O,A/2)

1.



$\rho = A$

2.

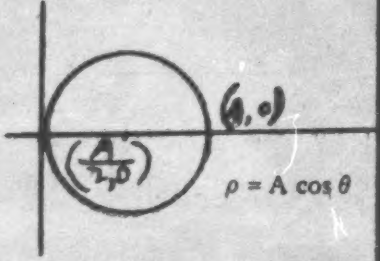


$\rho = A\cos\theta + B\sin\theta$

$$\text{RAD} = \frac{A^2 + B^2}{2}$$

$$\text{Center} = (\frac{A}{2}, \frac{B}{2})$$

3.



$\rho = A\cos\theta$

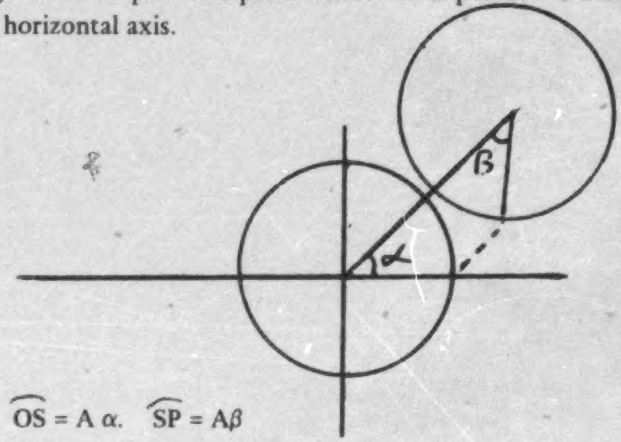
In Polar Coordinates, all conic sections can be written in the form:

$$R = \frac{eA}{1 + e\cos(\theta + \alpha)} \quad \text{or} \quad \frac{eA}{1 + e\sin(\theta + \beta)}$$

If $e = \pm 1$, it is a parabola; if $e < 1$, it is an ellipse; if $e > 1$, a hyperbola. $-\alpha$ and $-\beta$ are the angles between the principal axes and the X or Y axes.

Problem: Polar coordinates can be used in the following problem:

Given: two gears of equal size – one fixed and the other moving around it with no slipping. Trace the path of a point whose initial position is the point of contact at the horizontal axis.



$\overset{\frown}{OS} = \overset{\frown}{PS}$. $\overset{\frown}{OS} = A\alpha$. $\overset{\frown}{SP} = A\beta$

$\therefore \alpha = \beta$. $OQ = RP = A$, hence we have an equilateral trapezoid and $OP \parallel QR$.

$\therefore \theta = \alpha = \beta$.

The X coordinate of $P = \rho\cos\theta = -OQ + QR\cos\theta + RP\cos\phi$.

$\therefore \rho\cos\theta = -A + 2X\cos^{\theta}\theta + A\cos\phi$.

But $\theta + \phi + \theta = \pi$, and $\phi = \pi - 2\theta$. $\cos\phi = \cos(\pi - 2\theta) = -\cos 2\theta$.

$\rho\cos\theta = -A + 2A\cos\theta - A\cos 2\theta = 2A\cos\theta - A(1 + \cos 2\theta) = 2A\cos\theta - 2A^2\cos^2\theta$

$\rho = 2A - 2A\cos\theta$, The Cardioid.

Problem: Theorem of Guldinius

The volume generated by revolving an area A about an axis at distance $\overline{X}$ from the center of area is:

$$v = 2\pi\overline{X}^{2} A$$

Proof: $V = 2\pi\int(Y_2 - Y_1)X\,dX$

$$M_Y = \int X(Y_2 - Y_1)\,dX = \overline{X}\int(Y_2 - Y_1)dX = \frac{V}{2\pi} = \overline{X}A,\ V = 2\pi\overline{X}A$$

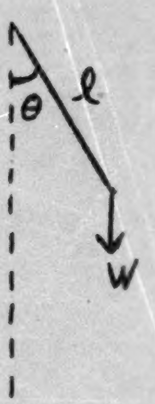
Example: Find the volume of a torus, whose radius is r, and where the distance from the center of rotation to the center of the circle is R.

$$V = 2\pi R\pi r^2 = 2\pi^2 R\,r^2$$

Example: Find the centroid of a semi-circle. $V = \frac{4}{3}R^3$

$$A = \frac{\pi R^2}{2} \qquad \frac{4}{3}R^3 = \frac{\overline{Y}\pi R^2}{2} \qquad \overline{Y} = \frac{4R}{3\pi}$$

Problem: Consider the simple pendulum.



$$W \sin\theta = \frac{-W}{g}\frac{d^2\theta}{dt^2} \qquad \text{or} \qquad \frac{d^2\theta}{dt^2} = \frac{-g}{\ell}\sin\theta$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \text{ etc., } \theta \text{ in radians.}$$

For θ small, $\sin\theta \approx \theta$, and $\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta$

Solutions:

(a) $\theta = A \sin\sqrt{\frac{g}{\ell}}\,t + B\cos\sqrt{\frac{g}{\ell}}\,t$

(b) Let $w = \frac{d\theta}{dt}$. Then $\frac{d^2\theta}{dt^2} = \frac{dw}{d\theta}\frac{d\theta}{dt} = w\frac{dw}{d\theta}$

Then $w\,dw = -\frac{g}{\ell}\theta\,d\theta$, and $w^2 = -\frac{g}{\ell}\theta^2 + \frac{g}{\ell}k^2 \quad w = \frac{d\theta}{dt} = \sqrt{\frac{g}{\ell}}\sqrt{k^2-\theta^2}$

$\sqrt{\frac{g}{\ell}}\,dt = \frac{d\theta}{\sqrt{k^2-\theta^2}}$, and $t\sqrt{\frac{g}{\ell}} = \text{arcs in}\,\frac{\theta}{k} + C$

$\theta = k\sin(\sqrt{\frac{g}{\ell}}\,t - C)$

(c) If the angle is large, the approximation of $\sin\theta = \theta$ cannot be used. One method of solution would be to use a computer to solve the D.E. Another would be as follows:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} = w\frac{dw}{dt} = -\frac{g}{\ell}\sin t$$

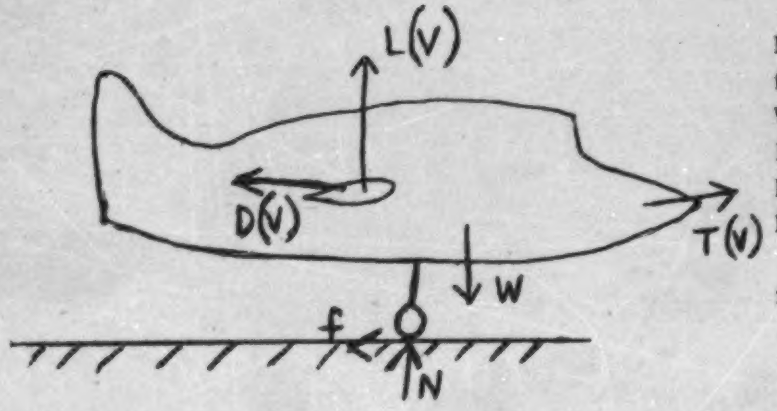
$$\frac{w^2}{2} = \frac{g\cos\theta}{\ell} + \frac{C}{2} \qquad w = \frac{dt}{d\theta} = \sqrt{\frac{g\cos t}{\ell}} + C$$

$$wdw = -\frac{g}{\ell}\sin\theta\,d\theta \qquad \frac{w^2}{2} = \frac{g}{\ell}\cos\theta + \frac{C}{2} \quad w = \sqrt{\frac{2g\cos\theta}{\ell}} + C = \frac{d\theta}{dt}$$

$$dt = \sqrt{\frac{d\theta}{\frac{2g\cos\theta}{\ell}} + C}$$

The second De can also be solved by integrating $\sqrt{\frac{d\theta}{\frac{2g\cos\theta}{\ell}} + C}$ using Trapezoidal or Simpson's Rule.

Problem: Find the runway distance an airplane requires in order to go from zero velocity to a given velocity, V.



$N = W - L$
$f = uN = u(W - L)$
W is the weight
L is the aerodynamic lift
D is the aerodynamic drag
N is the reaction force of the runway on wheels
f is the ground friction, and it is a fraction of N

$$T - D - f = \frac{W}{g}\frac{dV}{dt} = V\frac{dV}{dX}$$

Distance $= \displaystyle\int_0^V \frac{V\,dV}{T-D-f}$ L, T, D, and f are of the form $A + Bv^2$, Dis. $= \displaystyle\int_0^V \frac{Vdv}{E + FV^2}$

$$\text{Time} = \frac{g}{W}\int_0^V \frac{dV}{T-D-f} = \frac{g}{W}\int_0^V \frac{dv}{E + FV^2}$$

Problem: Compression or extension of a spring.

By Hooke's Law, we know that the force required to stretch or compress a spring from the equilibrium position is proportional to the change in length, or $F = kX$. Since work done is force times distance, the work done is $\int_{X_1}^{X_2} kX\,dX$. An integral must be used as the force is variable.

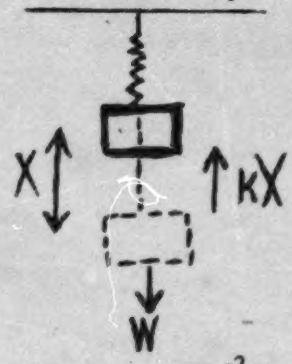
The work done is $\dfrac{k(X_2^2 - X_1^2)}{2}$

A typical problem would be to find the distance a bumper at the end of a train track would compress in stopping a moving train. Since the bumper is originally in equilibrium, $X_1 = 0$. The work done in compressing the spring must equal the original amount of kinetic energy possessed by the train; i.e.,

$\dfrac{mV^2}{2}$. Therefore, $\dfrac{mV^2}{2} = \dfrac{kX^2}{2}$, and $X = \sqrt{V\dfrac{m}{k}}$

Note, one must be careful to select compatible units.

Problem: Consider the problem of the vibration of a weight hanging from a spring. The weight is displaced a distance X from the equilibrium position and released.



For a small weight, we obtain $-kX = \frac{W}{g}\frac{d^2X}{dt^2}$

For a heavy weight, we obtain $-kX + W = \frac{W}{g}\frac{d^2X}{dt^2}$

By the usual methods of solving Linear Differential Equations, we obtain:

$X = A \sin\sqrt{\frac{gk}{W}}\,t + B \cos\sqrt{\frac{gk}{W}}\,t + (\frac{W}{k})$, for the heavy weight, and

$X = A \sin\sqrt{\frac{gk}{W}}\,t + B \cos\sqrt{\frac{kg}{W}}\,t$ for the light weight.

If these techniques are not available, consider the following approach.

If these techniques are not available, consider the following approach.

1. As the body goes up or down, the spring force will increase, slowing the the body to rest.
2. At that point, the unbalanced force will reverse the body until it reaches a resting point at the other extreme.
3. It will then reverse again, etc. etc. etc.
4. Functions which behave this way are sines and cosines.
5. Try X = A cos t. When this does not work, adjust it to X = a cos bt, and find a value for b. Similarly for sin bt.

The addition of a friction force proportional to velocity leads to:

$\frac{W}{g}\frac{d^2X}{dt^2} = -kX - c\frac{dX}{dt} + W$, with solution

$X = \mathrm{Exp}\left(-\frac{cg}{2W}\right)(A \cos rt + B \sin rt)$ where $r = ½\sqrt{\frac{c^2g^2}{W^2} - \frac{4kg}{W}}$

or X = a Exp (rt) b Exp (−rt) depending upon the nature of r. The solution is damped oscillatory motion, or damped non-oscillatory motion.

Problem: Snell's Law: Consider the following problem.

Find the path from point P_1 to P_2 where the velocity from P_1 to P_3 is V_1 and the velocity from P_3 to P_2 is V_2 and it is desired to find the path requiring the least time. This will not be a straight line. The time for each part of the trip will equal the distance divided by the corresponding velocity.

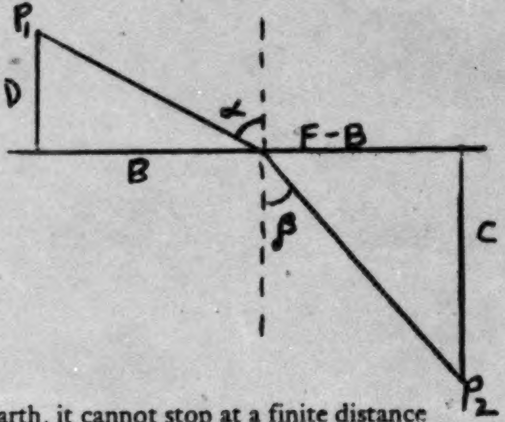
$$T = \frac{\sqrt{D^2 + B^2}}{V_1} + \frac{\sqrt{C^2 + (F - B)^2}}{V_2}$$

$$\frac{dT}{dB} = \frac{B}{V_1\sqrt{D^2 + B^2}} + \frac{F - B}{V_2\sqrt{C^2 + (F - B)^2}} = 0$$

$$\therefore \frac{B}{V_1\sqrt{D^2 + B^2}} = \frac{+B - F}{V_2\sqrt{C^2 + (F - B)^2}}$$

But $\dfrac{B}{\sqrt{D^2 + B^2}} = \sin\alpha$, and $\dfrac{-F + B}{\sqrt{C^2 + (F - B)^2}} = \sin\beta$

$$\therefore \frac{\sin\alpha}{V_1} = \frac{\sin\beta}{V_2}$$



Problem: Escape Velocity

In order for an object to escape from the earth, it cannot stop at a finite distance from the earth, since the gravitational force of the earth will pull the object back. The rest position must be at an infinite distance from the center of the earth. Let the mass of the earth be m_e and the mass of the object be m_o and the distance between their centers be s. Then:

$$-F = \frac{k\, m_e\, m_o}{s^2} = m_o \frac{dv}{dt} = m_o v \frac{dv}{ds} \qquad -v\,dv = \frac{k\, m_o\, ds}{s^2}$$

$$-\int_{v_o}^{0} v\,dv = \int_{R}^{\infty} k\, m_o\ s^{-2} ds = \lim_{h\to\infty} k\int_{R}^{h} m_o\ s^{-2}\,ds$$

where v_o is the velocity leaving the earth, and R is the radius of the earth.

$$\frac{-v^2}{2}\Bigg|_{v_o}^{0} = \lim_{h\to\infty} (-km_e s^{-1}\Bigg|_{e}^{h} \qquad \frac{-v^2}{2}$$

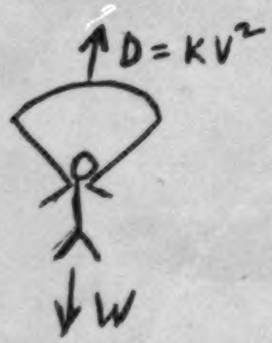
$$-0 + \frac{v_o^{\,2}}{2} = \lim_{h\to\infty} \left(\frac{-km_e}{h} - \frac{-km_e}{R}\right) = \frac{km_e}{R}$$

$$v_o^{\,2} = \frac{2km_e}{R}$$

$$v_o = \sqrt{\frac{2km_e}{R}}$$

Problem: Parachute Problem.

A body falls from rest (i.e.,$v_o = 0.$) with gravity and drag force, $d = kv^2$ acting upon it. The problem is to find the velocity and distance travelled at any time, t.



$$W - k^2 v^2 = \frac{W}{g} \frac{dv}{dt}$$

$$dt = \frac{W}{g} \frac{dv}{W - k^2 v^2}$$

$$t = \frac{\sqrt{W}}{2g\sqrt{k}} \text{LN}\left(\frac{1 + \frac{kv}{\sqrt{W}}}{1 - \frac{kv}{\sqrt{W}}} \right)$$

$$v = \frac{dx}{dt} \frac{W}{2k} \frac{e^{at} - 1}{e^{at} + 1}, \text{ where } a = \frac{2g\sqrt{k}}{\sqrt{W}}$$

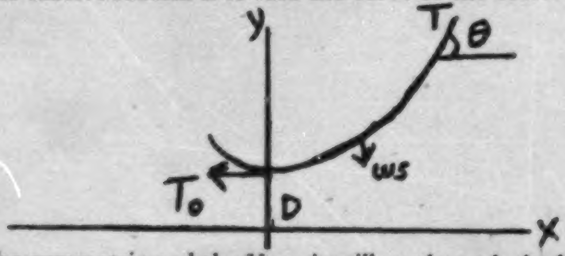
Note: As $t \to \infty$, $v \to \frac{\sqrt{W}}{k}$.

$$x = \frac{W}{k^2 g} \text{LN} (e^{bt} + e^{-bt}) + C, \text{ where } b = \frac{gk}{\sqrt{W}}$$

If for t = 0, x = 0, then C = 0.

Problem: The Catenary

The catenary is a freely hanging chain or cable perfectly flexible and hanging on its own weight. Because of the flexibility, there are no side loads and all forces are *tangential to the cable. Problem is to find the curve of the cable.*



The cable will be symmetric and the Y–axis will go through the lowest point. The X–axis will be at a distance D below this point. We will select D later.

The linear density of the cable is w lb/ft and the cable is s ft long. Weight is assumed to act at the cg of a section.

From the conditions of equilibrium, we know:

$$T_o = T\cos\theta \qquad ws = T\sin\theta$$

$$\frac{ws}{T_o} = \frac{T\sin\theta}{T\cos\theta} = \tan\theta = \frac{dy}{dx}$$

Taking the derivative of both sides, we get:

$$\frac{d^2y}{dx^2} = \frac{w}{T_o}\,\frac{ds}{dx}, \text{ but s is arc length, and } \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{w}{T_o}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Let $v = \frac{dy}{dx}$, with $v = 0$, when $x = 0$.

$$\frac{dv}{dx} = \frac{w}{T_o}\sqrt{1+v^2}$$

$$\text{Ln}\,(v+\sqrt{v^2+1}\,) = \frac{wx}{T_o} + C$$

$$\text{Ln}\,1 = 0 + C,\quad C = 0$$

$$v + \sqrt{v^2+1} = e^{wx/T_o}$$

$$v = \frac{dy}{dx} = \tfrac{1}{2}\cdot e^{wx/T_o} - \tfrac{1}{2}\cdot e^{-wx/T_o}$$

$$y = \frac{T_o}{2w}(e^{wx/T_o} + e^{-wx/T_o}) + C$$

Let $D = \frac{T_o}{w}$, $x = 0$, $y = \frac{T_o}{w}$, and $C_1 = 0$.

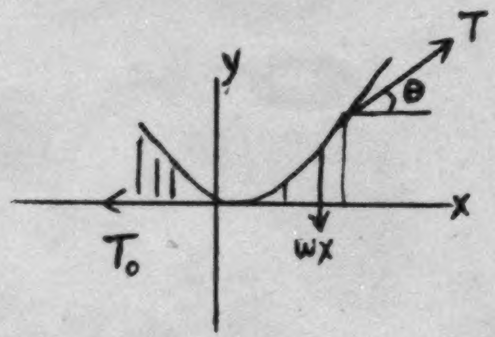
$$y = \frac{T_o}{2w}(e^{wx/T_o} + e^{-wx/T_o})$$

If we wished to introduce hyperbolic functions at this point, we could write the above as:

$$y = \frac{T_o}{2w}\,\text{Cos}\ \frac{wx}{T_o}$$

Problem: Bridge Hanging from a Flexible Cable

In this problem, we assume that the bridge has a fixed weight per length and that the bridge is much heavier than the cable, so that the cable weight can be ignored. We will select the origin at the lowest point of the cable.



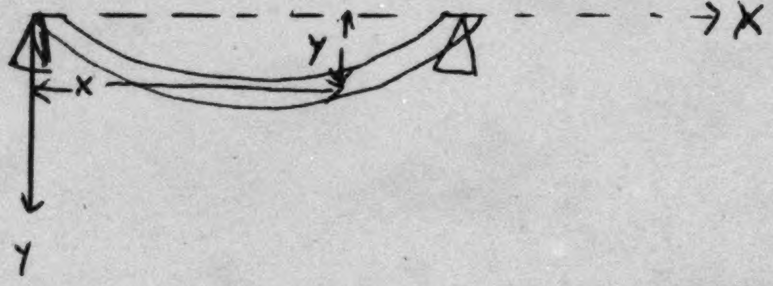
$T_o = T\cos\theta \qquad wx = T\sin\theta$

$$\frac{wx}{T_o} = \frac{T\sin\theta}{T\cos\theta} = \tan\theta = \frac{dy}{dx}$$

$$y = \frac{wx^2}{2T_o} + C_1 \quad x = 0,\ y = 0,\ \text{so } C_1 = 0$$

$$y = \frac{wx^2}{T_o}$$

Problem: Deflection of Beams



The deflection of a beam is governed by the differential equation, $EIy''' = w(x)$, where I is the moment of inertia of a cross-section and could be variable, E is the modulus of elasticity and is a property of the material used, and w(x) is the load. Note y'' is the moment at any point, and will be zero at the ends, if they are free.

Consider an unloaded beam, bending solely due to its own weight which is distributed evenly with density w lb/ft, and simply supported at the ends; i.e., y and y'' are zero at both ends. Assume E and I constant and length L.

$$EIy''' = wx \quad \text{and} \quad EIy'' = \frac{wx^2}{2} + C_1 x + C_2$$

$$EIy' = \frac{wx^3}{6} + \frac{C_1 x^2}{2} + C_2 x + C_3$$

$$EIy = \frac{wx^4}{24} + \frac{C_1 x^3}{6} + \frac{C_2 x^2}{2} + C_3 x + C_4$$

$$x = 0, \ y = 0, \ y'' = 0. \quad x = 1, \ y = 0, \ y'' = 0.$$

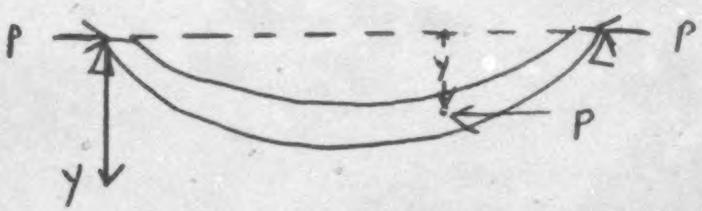
$$0 = C_4. \quad 0 = C_2.$$

$$0 = \frac{w\ell^4}{24} + \frac{C_1 \ell^3}{6} + \frac{C_2 \ell^2}{2} + C_3 1 + C_4$$

$$0 = \frac{w\ell^2}{2} - C_1{}^1$$

$$C_1 = \frac{w\ell}{2}. \quad C_3 = \frac{w\ell^3}{24}. \qquad y = \frac{wx^4}{24} - \frac{w\ell}{12}x^3 - \frac{w\ell^3}{24}x$$

Another beam deflection with some interesting mathematical aspects is that of a beam under compressive forces at the ends. The weight of the beam is ignored and it is assumed that the beam does not crush.



Moments about the left support gives:

$$EIy'' = P(O) - P(y) \quad \text{or} \quad EIy'' + Py = 0 \quad \text{or} \quad y'' + \frac{P}{EI}y = 0, \text{ with}$$

$x = 0,\ y = 0,$ and $x = 1,\ y = 0.$

$$\therefore\ y = A \sin\sqrt{\frac{P}{EI}}\,x + B \cos\sqrt{\frac{P}{EI}}\,x$$

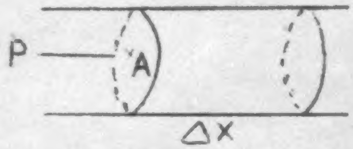
The given conditions lead to: $0 = B$ and $0 = A \sin\sqrt{\frac{P}{EI}}\,\ell$

For the second condition to be satisfied, either $A = 0$ and the beam does not deflect at all, or:

$$\sqrt{\frac{P}{EI}} = \pi \quad \text{and} \quad P = \frac{\pi^2 EI}{\ell^2}, \ \frac{4\pi^2 EI}{\ell^2}, \ldots, \frac{n^2\pi^2 EI}{\ell^2}$$

Notice that A is not determined, which means that when the beam buckles, the amount of buckling cannot be determined, so that in all actuality it has failed. Furthermore, the beam will buckle only at the loads indicated; i.e., if it does not fail at $P = \frac{\pi^2 EI}{\ell^2}$, it will not fail at higher loads less than $P = \frac{4\pi^2 EI}{\ell^2}$. However, it is obvious that we cannot assume the beam will survive a load of $P = \frac{\pi^2 EI}{\ell^2}$, which is therefore listed as the critical load.

Problem: Work is done in expanding a gas. Work is generally considered as Force times Distance. Consider an element of gas in a circular cylinder, with radius constant over a small interval. The force is pressure X Area. Distance moved is Δx.



Work = $P\,A\,\Delta x$. But $A\,\Delta x = \Delta V$ (Volume)
Work = $\Sigma P\,\Delta V$ $W = \int P\,dV$

There are two special cases governing expansion of gases. One is isothermal (Constant temperature) where $PV = C$, and the other adiabatic (Constant heat, which can occur in rapid changes lacking time for heat transfer), $PV^{\gamma} = K$

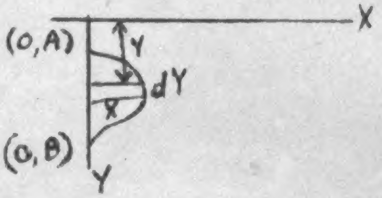
$$\text{Isothermal: } W = \int_{V_1}^{V_2} P\,dV = \int_{V_1}^{V_2} CV^{-1}dV = C\,\text{Ln}\,\frac{V_2}{V_1}$$

$$\text{Adiabatic: } W = \int_{V_1}^{V_2} P\,dV = \int_{V_1}^{V_2} K\,V^{-\gamma}dv = \frac{K\,V_2^{\,1-\gamma} - K\,V_1^{\,1-\gamma}}{1-\gamma} = \frac{P_2V_2 - P_1V_1}{1-\gamma}$$

Note: γ usually takes on the value of 1.4 or 1.66 depending on the gas.

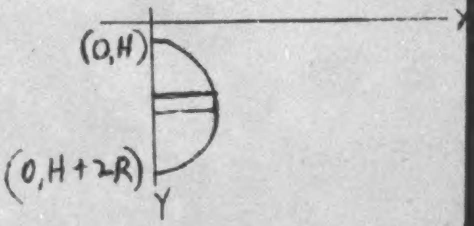
Problem: Force of a liquid on a submerged vertical plate.

Force in a liquid is equal to pressure times area. Since pressure varies with depth, we must calculate force at a fixed depth, and then sum this from the top to bottom of the plate; i.e., integrate.



$F = \int_A^B d\ Y\ X\ dY$, for compressible fluids, d is a variable. For incompressible fluids, d is constant.

A. Find the total force on a semi-circular plate of radius R, whose top is H feet below the surface of an incompressible fluid.



$$F = d\int_H^{H+2R} XY\ dY,\ X^2 + (Y - H - R)^2 = R^2$$

$$= d\int_H^{H+2R} (R^2 - (Y - H - R)^2)^{1/2}\ ydY$$

Let $Y - H - R = R \sin\theta \quad dY = R\cos\theta d\theta$

$\Delta \quad Y = H + R + R \sin\theta$

$$F = d\int_{-\pi/2}^{\pi/2} (H + R + R\sin\theta)R\cos\theta R\ \cos\theta d\theta$$

$$= dR^2\int_{-\pi/2}^{\pi/2} (\ (H + R)\cos^2\theta\ +\ R\sin\theta\cos^2\theta)\ d\theta$$

$$= dR^2\ (\ (H + R)\ (\frac{\theta}{2} + \frac{\sin 2\theta}{4}\ -\ \frac{R\cos^3\theta}{3})\Big/_{-\pi/2}^{\pi/2}$$

$$= dR^2\ (\ (H + R)\ (\frac{\pi}{4})\ -\ (H + R)\ (-\frac{\pi}{4})$$

$$= \frac{\pi dR^2\ (H + R)}{2}$$

Problem: Center of pressure for a submerged vertical plate.

The Center of Pressure is that point where the total force can be considered to be acting and which would give the same moment as the distributed forces.

$$\bar{Y}\int_{A}^{B} dXY\,dY = \int_{A}^{B} dXY^2 dY, \quad \text{or} \quad \bar{Y} = \frac{\int_{A}^{B} dXY^2\,dY}{\int_{A}^{B} dXY\,dY}$$

For a rectangular plate of width w and height H whose top is k feet below the surface, assume incompressibility.

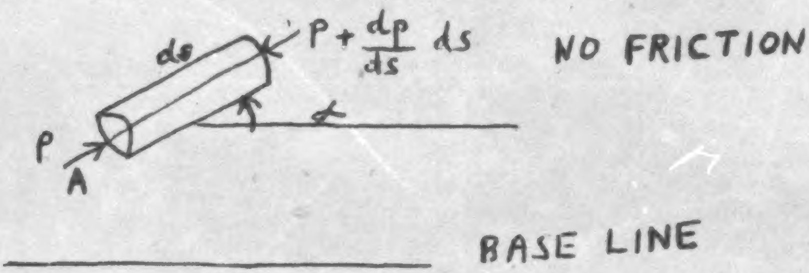
$$Y = \frac{d\,HY^2\,dY}{d\,HY\,dY} = \frac{dHY^{\frac{3}{3}}\Big|_{k}^{k+H}}{dH\,Y^{\frac{2}{2}}\Big|_{k}^{k+H}} = \frac{2(3k^2 + 3kH + H^2)}{3\,(2k + H)}$$

The force is $\dfrac{dH\,(2kH + H^2)}{2}$

The moment is $\dfrac{dH\,(3k^2H + 3kH^2 + H^3)}{3}$

Problem: Derivation of Bernoulli's Equation.

Consider the flow of a small piece of fluid along a stream-line (velocity tangent to it). Using F = ma, we will derive the equation for both incompressible and compressible flow.



mass = d A ds

If velocity = u, then $a = \frac{du}{dt} = \frac{duds}{dsdt} = u\frac{du}{ds}$

Then, $-A\left(P + \frac{dP}{ds}\,ds\right) + AP - dAg\sin\alpha\, ds = dAu\frac{du}{ds}\,ds$

pressure force:

$$\frac{udu}{ds} = -\frac{1}{d}\frac{dP}{ds} - g\frac{dh}{ds}$$

gravity force:

$$\sin\alpha = \frac{dh}{ds}$$

$$\frac{1}{d}\frac{dP}{ds} + u\frac{du}{ds} + g\frac{dh}{ds} = 0$$

$$\int_{P_1}^{P_2}\frac{dP}{d} + \int_{u_1}^{u_2} u\,du + g\int_{h_1}^{h_2} dh = 0$$

if d is constant, incompressible, we obtain:

$$\frac{P_1}{d} + \frac{{u_1}^2}{2} + gh_1 = \frac{P_2}{d} + \frac{{u_2}^2}{2} + gh_2$$

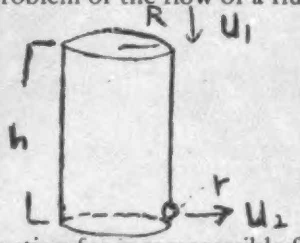
For compressible, isentropic (frictionless and reversible) flow, d and P are related by:

$$P = k\,d^{\gamma} \quad \text{or} \quad d = \frac{P^{1/\gamma}}{k^{1/\gamma}}$$

We obtain: $\int k^{1/\gamma} P^{-1/\gamma} dP + \frac{u^2}{2} + gh = C$ $\qquad \frac{k^{1/\gamma} P^{1-1/\gamma}}{1-\gamma} + \frac{u^2}{2} + gh = C$

$$\frac{\gamma P}{d(\gamma-1)} + \frac{u^2}{2} + gh = C$$

Problem: Consider the problem of the flow of a fluid from a large cylinder through a small hole.



Radius of cylinder = R

Radius of orifice = r

r << R

Bernoulli's Equation for incompressible fluids.

$$\frac{P_1}{d} + \frac{u_1^2}{2} + gh_1 = \frac{P_2}{d} + \frac{u_2^2}{2} + gh_2$$

$$h_2 = 0, \quad \pi u_1 R^2 = \pi u_2 r^2. \quad P_2 - P_1 = dgh$$

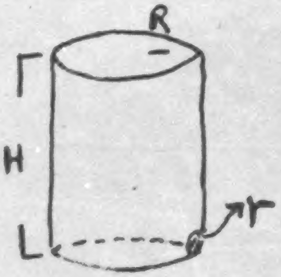
Since r << R, u_1 is small enough to be ignored, therefore:

$$\frac{P_2 - P_1}{d} = gh = \frac{u_2^2}{2} \qquad u_2 = \sqrt{2gh}$$

This is theoretical. For actual cases with water and a circular orifice with sharp edges:

$$u = 0.62\sqrt{2gh}$$

Cylinder Height H and Radius R



Vel. $v = kh^{1/2}$ Vol. $V = \pi R^2 h$

$$\frac{dV}{dt} = -\pi R^2 \frac{dh}{dt}$$

$$= \pi r^2 v = -\pi r^2 k h^{1/2}$$

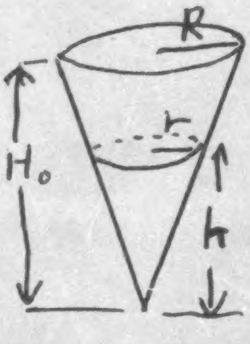
$$-h^{-1/2}\,dh = \frac{\pi r^2 k\,dt}{\pi R^2}$$

The time required for the height to drop from H_o to H is:

$$t = \frac{-R^2}{kr^2}\int_{H_0}^{H} h^{-1/2}\,dh \qquad t = \frac{2R^2}{kr^2}\left(\sqrt{H_o} - \sqrt{H}\right)$$

Time to lower the height of fluid in a cone from H_o to H. The height is H_o , the radius R.

$$\frac{r}{h} = \frac{R}{H_o} \qquad V = \frac{\pi r^2 h}{3} = \frac{\pi R^2 h^3}{3H^2}$$

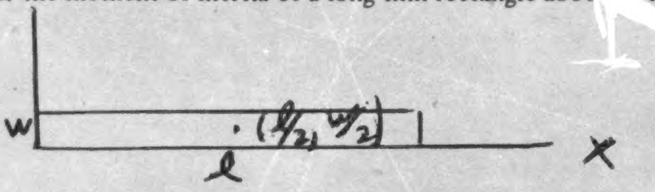


$$\frac{dV}{dt} = -\frac{\pi R^2 h^2}{\pi H_o^2 r_1^2 k h^{1/2}} \frac{dh}{dt}$$

$$dt = -\frac{R^2 h^{3/2}}{k H_o^2 r_1} \frac{dh}{2}$$

$$t = \frac{2R^2}{5kH_o^2 r_1^2}\left(H_o^{5/2} - H^{5/2}\right)$$

Problem: Consider the moment of inertia of a long thin rectangle abou[illegible]s long side, $w << \ell$.



$$I_X = I_{cg} + m\,(w/2)^2 \qquad I_X = \frac{m(w/2)^2}{12} + \frac{mw^2}{4} = \frac{mw^2}{48} + \frac{mw^2}{4}$$

We can almost ignore I_{cg}.

Problem: Newton's Law of Cooling

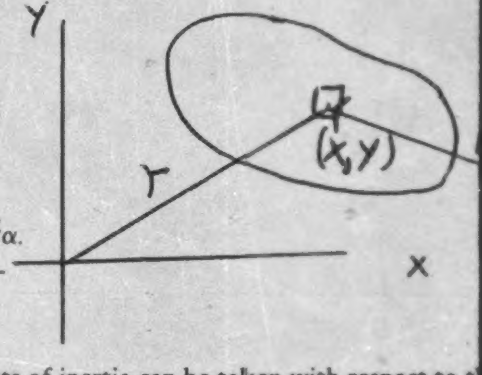
Newton's Law of Cooling states that the rate of cooling is directly proportional to the difference between the current temperature and the fixed temperature about the body T_o, or:

$$\frac{dT}{dt} = k\,(T - T_o) \qquad \int_{T_1}^{T_2} \frac{dT}{T - T_o} = k \int_{t_1}^{t_2} dt$$

$$\mathrm{Ln}\left(\frac{T_2 - T_o}{T_1 - T_o}\right) = k\,(t_2 - t_1) \qquad \frac{T_2 - T_o}{T_1 - T_o} = e^{k\,(t_2 - t_1)}$$

$$T_2 = T_o + (T_1 - T_o)\,e^{k(t_2 - t_1)}$$

Problem: Rotation



Newton's Law tells us that F = ma, but in rotational problems, we know that it is not force but moment that interests us. Also we will be interested in angular acceleration α rather than linear acceleration, a. The relations between these variables are:

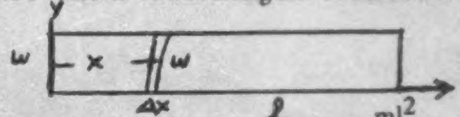
$M = F\,r$ and $a = r\alpha$

We can rewrite $F = ma$ as $rF = mar$ or $rF = mr^2\alpha$. It would seem reasonable to consider the quantity mr^2 as the rotational equivalent of linear motion mass. We shall call it the moment of inertia, usually symbolized by I. These moments of inertia can be taken with respect to t X axis, the Y axis, or the origin, and are labelled I_X, I_Y, I_o, respectively.

To find the moment of inertia of an entire body, we would sum up the moments of inertia of all the parts. If we proceed to the infinite sum in the usual way, we obtain the usual integral.

Moment of inertia is not only important in problems of rotation, but it is very important in beam deflection problems, where the shape, i.e., the distribution of the mass, is very important.

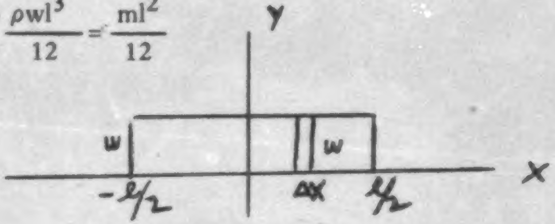
Consider the problem of finding the moment of inertia of a rectangular beam about one end.



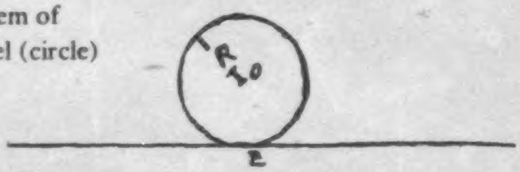
$$I_Y = \int_0^{\ell} \rho X^2\, w\, dX = \left.\frac{\rho\, wX^3}{3}\right|_0^{\ell} = \frac{\rho\, wl^3}{3}$$. But Mass $m = \rho\, wl$, and hence $I_Y = \frac{ml^2}{3}$

The center of gravity is obviously at the center. If we wish to obtain the moment of inertia about the center of gravity, I_{cg}, we obtain

$$I_{cg} = \int_{-\ell/2}^{\ell/2} X^2 w\, dX = \frac{\rho w X^3}{3}\Big|_{-\ell/2}^{\ell/2} = \frac{\rho w l^3}{12} = \frac{ml^2}{12}$$



Note that $I_Y = I_{cg} + \frac{ml^2}{4}$, or $I_Y = I_{cg} + md^2$, where d is the distance from the cg to the axis of interest. This is the well known transfer theorem, which is of great practical use. Consider the problem of computing the moment of inertia of a wheel (circle) about its point of contact.



$$I_o = \frac{mr^2}{2}. \quad I_z = \frac{mr^2}{2} + mr^2 = \frac{3mr^2}{2}$$

Note: The transfer theorem can only be used when transferring from the cg. Also since $I > 0$, I_{cg} is the minimal value of I.

There are many applications where polar coordinates are useful. These usually are problems involving circles with center at the origin or going through the origin, and certain motion problems.

Problem: Find the moment of inertia of a circle about its center (polar moment). In Cartesian Coordinates, one obtains:

$$4\int_0^A\int_0^{\sqrt{A^2+Y^2}}(X^2+Y^2)\,dY\,dX = 4\int_0^A\left(X^2\sqrt{A^2-X^2}+\frac{(A^2-X^2)^{3/2}}{3}\right)dX$$

Let $X = A\sin\theta$, $\quad dX = A\cos\theta\,d\theta$

$$4\int_0^{\pi/2}\left(A^3\sin^2\theta\cos\theta+\frac{A^3\cos^3\theta}{3}\right)A\cos\theta\,d\theta,\quad\text{or}$$

$$4A^4\int_0^{\pi/2}\left(\sin^2\theta\cos^2\theta+\frac{\cos^4\theta}{3}\right)d\theta = 4A^4\int_0^{\pi/2}\left(\frac{\sin^2 2\theta}{4}+\frac{1}{12}(I+2\cos 2\theta+\cos^2 2\theta)\right)d$$

$$4A^4\left(¼\left(\frac{\theta}{2}-\frac{\sin 4\theta}{8}\right)+\frac{1}{12}\left(\theta+\sin 2\theta+\frac{\theta}{2}+\sin\frac{4\theta}{4}\right)\Big/_{0}^{\pi/2} = \frac{\pi A^4}{2}\right.$$

If we equate mass to area, then $M = \text{Area} = \pi A^2$, and $I_o = \dfrac{M A^2}{2}$.

In polar coordinates^2, the element of area is $\rho d\rho d\theta$ and the integral for polar moment is :

$$\int_0^{2\pi}\int_0^{A}\rho^2\rho\,d\rho\,d\theta = \int_0^{2\pi}\frac{\rho^4}{4}\Big|_0^{A}d\theta = \frac{A^4}{4}\theta\Big/_0^{2\pi} = \frac{\pi A^4}{2}$$

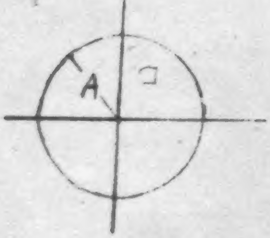
For a hollow ring^3 of inner radius B, and outer radius A, we obtain:

$$\int_0^{2\pi}\int_B^{A}\rho^3\,d\rho\,d\theta = \int_0^{2\pi}\frac{\rho^4}{4}\Big|_B^{A}d\theta = \frac{A^4-B^4}{4}\theta\Big/_0^{2\pi} = \frac{\pi(A^4-B^4)}{2} = \frac{\pi(A^2-B^2)(A^2+B^2)}{2}$$

But $\pi\dfrac{A^2-B^2}{2}$ is the area of mass of the ring, hence $I_o = \dfrac{M(A^2+B^2)}{2}$

If we consider a thin ring where $B \approx A$, we obtain $I_o = \dfrac{M(A^2+A^2)}{2} = MA^2$.

1.



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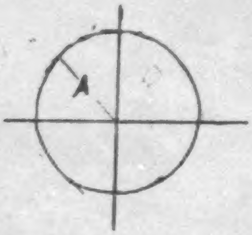

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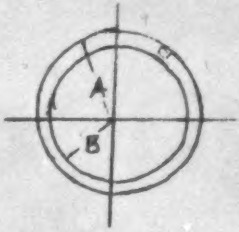
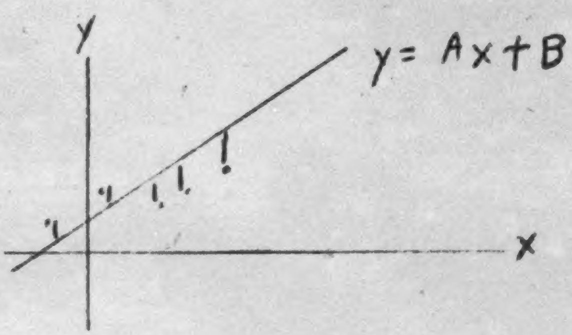


Problem: Curve Fitting – Best Straight Line

The object is to take a set of points which lie about a straight line, and find the best line. By this is meant the line, such that the sum of the distances squared from points to line is a minimum.



We will use subscripts to indicate observed points, and omit them for points on the line. We wish to minimize the sum of $(y_o - y)^2$, or minimize $\Sigma(y_o - Ax - B)^2$.

We will find the minimum by taking the partial derivatives with respect to A and B, and setting them equal to zero.

$$-\Sigma\, 2x\,(y_o - Ax - B) = 0$$

$$-\Sigma\, 2\,(y_o - Ax - B) = 0 \qquad \text{or}$$

$$A\,\Sigma\, x^2 + B\,\Sigma\, x = \Sigma xy$$

$$A\,\Sigma\, x + B\,n = \Sigma y$$

$$A = \frac{\begin{vmatrix} \Sigma xy & \Sigma x \\ \Sigma y & n \end{vmatrix}}{\begin{vmatrix} \Sigma x^2 & \Sigma x \\ \Sigma x & n \end{vmatrix}} \qquad B = \frac{\begin{vmatrix} \Sigma x^2 & \Sigma xy \\ \Sigma x & \Sigma y \end{vmatrix}}{\begin{vmatrix} \Sigma x^2 & \Sigma x \\ \Sigma x & n \end{vmatrix}}$$

TECHNICAL PHYSICS

I. Problems Involving Proportionality

A number of laws in physics can be stated in simple algebraic terms. It is important for the students to understand the implications of the laws and also how such laws (or rules) can be derived.

(a) The simplest case is the purely linear law such as

$F = ma$ (Eq. 1) or $V = IR$ (Eq. 2)

(Eq. 1) is the mathematical expression of Newton's second law, namely that, if a force "F" acts on a body of mass, "m", this mass accelerates with acceleration "a". It should be noted first that the law really states that the force F is **proportional** to "a", with the factor of proportionality being the number m, called the "inertial mass" and which clearly is different for each object.

What should be emphasized to the student is that doubling the force results in doubling the acceleration; halving the force, halves the acceleration. The same procedure can be used with equation (2), Ohm's law. It relates the voltage across a circuit to the current (I). R is the resistance of the wire (or circuit). Again, the current is proportional to the voltage, assuming no changes in the wire. For example if R is 10 ohms and V = 50 volts, then $50 = 10I$, $I = 5$ amperes.

(b) The more complicated law is when the proportionality involves higher powers. For instance:

The heat developed in a wire in which a current "I" exists is given by:

Heat $= I^2R$ (Eq. 3), where R is the resistance in the wire.

Here the emphasis should be on the fact that heat goes up as the **square** of the current. Hence increasing the current is more "effective" than changing the resistance. One can point out that physically (or practically), the reason why fuses can blow (a fuse is really only a thin wire placed in the circuit) when one starts the air conditioner is that initially the current is high, so that the heat developed (I^2R) melts the fuse wire.

(c) Fourth power proportionality.

Most people nowadays are conscious of cholesterol problems. The reason cholesterol accumulation in arteries and veins is important is because of the narrowness of passages that it provokes. Why should this fact be so dangerous to one's health? The reason lies in the relationship between pressure, speed of flow and tube radius.

If a liquid is flowing in a tube, the drop in pressure between the ends of the tube is given by the following equation:

$$P_1 - P_2 = \frac{8nL}{\pi r^4} F$$

Where P_1 is pressure at point 1, P_2 pressure at point 2; n is the viscosity of the liquid; r is the radius of the tube, L its length and F is the flow velocity.

If one were to apply this to blood flow, one could reason as follows:

The flow F must stay the same, otherwise the nutrients carried by the blood would not reach the organs in time.

L and n do not change since neither is affected much by cholesterol.

But r, the radius of the vein or artery, is reduced by cholesterol. If r is halved in such a case, the $P_1 - P_2$ must increase by a factor of 16 (2^4). This means that the heart would have to work 16 times as much and . . . .

(d) Square roots

If an object is dropped from rest, then assuming negligible air resistance, the velocity of the object after having traveled a distance d is given by:

$v^2 = 2ad$ where a is the acceleration

$v = \sqrt{2ad}$

Emphasis should be placed on the fact that the velocity does not increase as fast as the distance.

(e) Increase square law.

The two most famous examples are the law of gravitation and the law of electrostatic force.

(1) If two masses m_1 and m_2 are a distance "d" apart, the force of attraction between them is given by:

$$F = g \frac{m_1 m_2}{d^2}$$

where g is the universal constant of gravitation.

(2) If two charges q_1 and q_2 are a distance "d" apart, the force on each of them (attraction if one is positive and the other charge negative, repulsion if the charges are of same polarity), is given by:

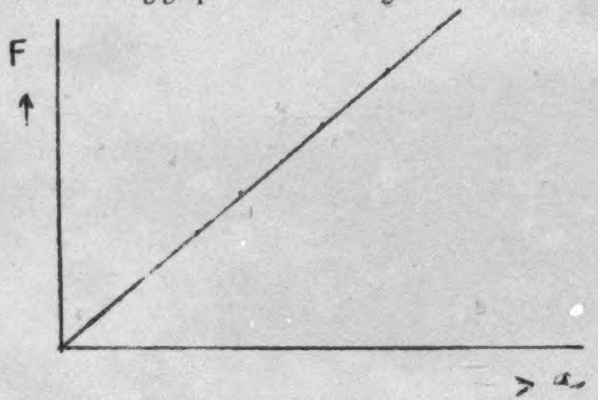
$$F = \frac{q_1 q_2}{d^2}$$

Emphasis is to be placed on the fact that if, e.g., the distance is doubled, the force is 4 times smaller, etc.

II. Graphs

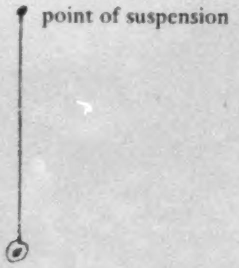
At least for the case of simple proportionality ($F = ma$ or $V = IR$), it is possible and useful to show the student how graphing can be utilized either to obtain the formula or represent it.

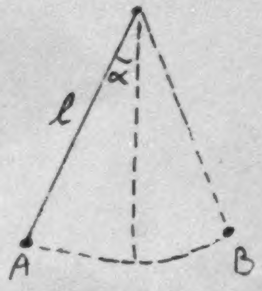
(1) The first illustration is the equation $F = ma$. One chooses two perpendicular axes, measures F along the vertical axis and "a" along the horizontal axis. For constant "m", meaning if one applies forces of varying size to the same mass, the resulting acceleration should be given by $F = ma$. The resulting graph is then a straight line.



Of course this must be so since the equation represents a straight line. If one did not know the law, a series of experiments with forces of varying size would produce, when plotted, the same graph

(2) A more complex example is the following: Consider the case of the period of a pendulum. A pendulum is composed of a string (or rigid bar) with a bob (ball) at one end and suspended at the other end.





It can be made to oscillate back and forth and the time it takes to return to its original position (from A to B and back to A) is called the period. The possible variables are, the length 1 of the string, the mass of the bob, the angle α . It is easy to show experimentally that

$$T = 2\pi \frac{\ell}{g}$$

where T is the period, ℓ is the length of the pendulum and g is the acceleration due to gravity

$$\left(9.8 \frac{m}{sec^2} \quad \text{or} \quad 32 \frac{ft}{sec^2}\right)$$

Since both 2π and g are constants, this expression can be rewritten as $T = A\ \ell$. Again, it should be emphasized to the student that if one plots T versus $\sqrt{\ell}$, a straight line will result.

Alternatively, one could give the student a series of experimental results using one type of pendulum; first change the mass of the bob and the resulting periods with different lengths. Here obviously the period changes. The students are then asked to plot period vs. length, period vs. ℓ^2, period vs. $\sqrt{\ell}$; the correct dependence is then the graph that produces a straight line relationship.

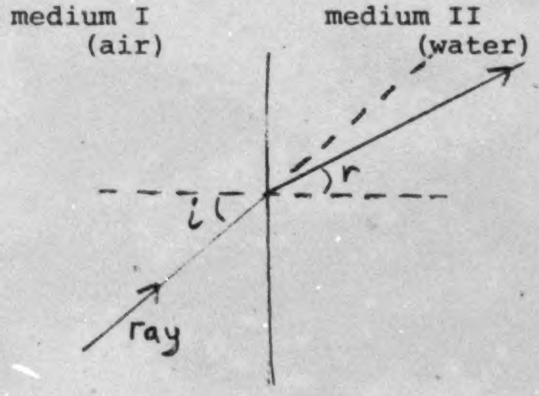
III. Trigonometry and Angles.

After a definition of angles (both in degrees and radians), one should introduce the trigonometric functions by pointing out that these are intrinsic rather than arbitrary angular units.

There are two main types of applications of trigonometry in physics. The first ones are those I would call direct application, the others are indirect applications occurring because they involve vectors. An example of the first kind is Snell's Law.

A. Snell's Law.

It is well known that when a ray of light (actually any electromagnetic wave) is refracted, meaning that it travels from one medium to another, it changes direction. Typically it looks as follows:



The ray travels from medium I into medium II. The angle the incident ray makes with the normal to the boundary between the two media is $\hat{i}$ and the angle made by the refracted ray with same normal is $\hat{r}$. Each medium is characterized by its index of refraction, defined as ratio of the speed of light in vacuum to the speed of light in the medium and indicated by the letter "u".

The relation between $\hat{i}$ and $\hat{r}$ is then given by Snell's Law.

$u_i \sin \hat{i} = u_r \sin \hat{r}$

where u_i is the index of the medium of the incident ray and u_r is index of the medium of the refracted ray.

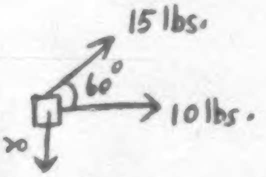
B. Vectors

More general applications of trigonometry involve vectors. As is well known, a vector is a quantity requiring both size and direction for its complete specification. It is assumed that the reader is familiar with the mathematics of such concepts and hence only applications will be given here.

Examples of vectors in physics are: force, velocity, acceleration, electric and magnetic fields, etc.

1. Applications involving only addition and subtraction of vectors.
   It may be easier to introduce the whole problem by using two practical examples as follows:

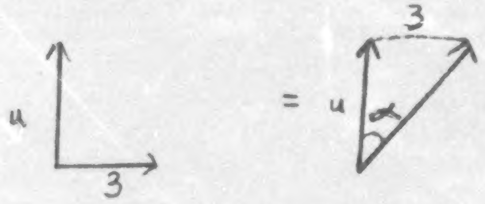
   An object weighing 20 lbs. is pushed to the right by Jack, exerting a force of 10 lbs., and at the same time pushed with a force of 15lbs. in a direction of 60° north of right by John.



We would like to know in what direction and with what strength an "equivalent force" would be exerted. This is the same as asking in what direction the box would move (and with how much acceleration).

A similar class of problems can be devised using velocities. Suppose a boat is heading north in a river with speed of 4 knots/hour when the current is eastward with speed of 3 knots/hour. What is the speed of the boat and in which direction is it moving?

(a) In summing two vectors which are perpendicular to each other, one has to add two vectors.

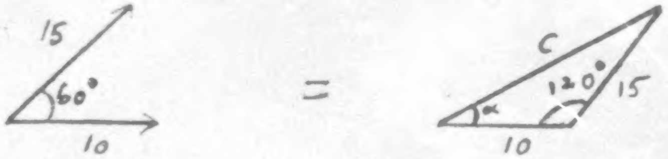


The Pythagorean theorem tells us immediately that the total velocity of the boat (its velocity with respect to shore) is 5 knots.

The angle with the north direction, as shown in figure (α), is given by $\tan \alpha = \frac{3}{4} = .75$. Hence, α is 37°.

We have here the first application of trigonometry. Emphasis should be placed on the fact that, although this result could have been obtained graphically, it is faster and much more accurate analytically.

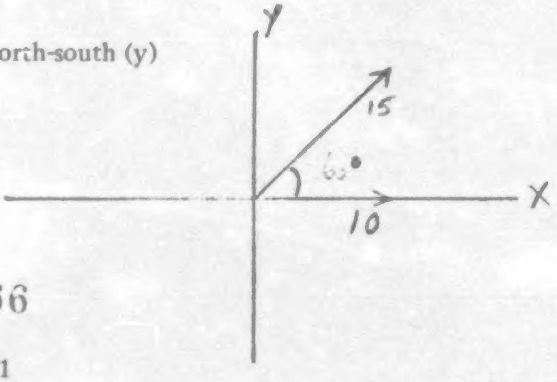
(b) Addition of non-parallel vectors.
Consider the problem of the two forces:



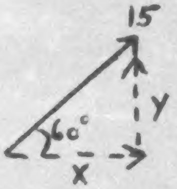
The length of C can of course be obtained by the law of cosines and then the angle α by the law of sines. This is a cumbersome method and not advisable when summing more than two vectors. The easier method is to reduce this problem to the preceding one by using the "components" method.

One first explains that since the sum of i vectors is another vector, any vector can be made to be the sum of 2 vectors, perpendicular to each other. By now taking 2 perpendicular axes, we will decompose the 2 vectors we want to add in sum of vectors always perpendicular to each other.

Let one direction be east-west (x), the other north-south (y)

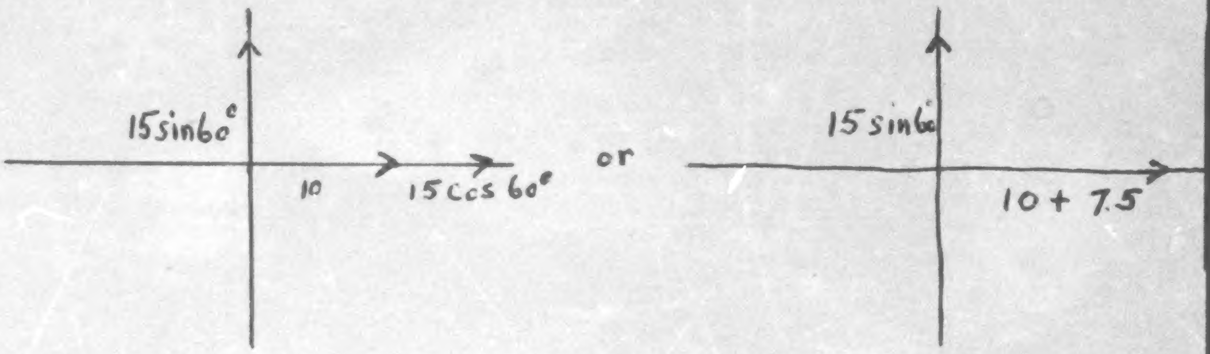


The first vector (10) is already in the x direction. Vector 15 can be considered as the sum of 2 perpendicular vectors as shown.



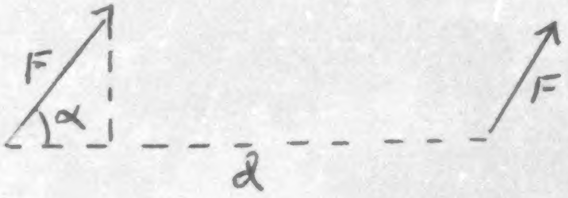
where x is 15 cos 60 y is 15 sin 60
and now we have 2 vectors (A and B)

Now we are back in case 1, the sum of 2 perpendicular vectors.



2. Another simple application of trigonometry in physics is in the concept of work.

Work in physics is the product of the force by the displacement of this force, but only the displacement in the direction of the force.



This is equivalent to saying that $W = F_{\alpha} \times d = Fd \cos \alpha$
or, in words, the work done is equal to the displacement of the force times the projection of the force in the direction of the displacement.

As an interesting illustration, one might point out that if the force is perpendicular to the displacement, no work (in physics sense) is performed.

IV. Units, Constants and Conversions

Clearly, since physics is basically an experimental science, systems of units, conversions from one system to the other and constants are of prime importance.

1. Units

Two important points must be emphasized to students about units. The first is that all formulas and equations must be dimensionally correct; and, secondly, that one can often derive missing terms of the equations by dimensional analysis.

The basic units are length, mass and time. Although it is common to stress the importance of the English system of units, one must now take into account the fact that we probably will "go metric" pretty soon.

English System	**Metric System**
Length: foot	Length: meter (centimeter)
Time: second	Time: second
Weight: pound	Mass: kilogram (gram)

The derived units must then comply dimensionally. This means, in some sense, that the unit symbols can be used as algebraic symbols.

$$\text{Example: Velocity} = \frac{\text{distance}}{\text{time}} = \frac{\text{ft}}{\text{sec}} \text{ or } \frac{\text{meter}}{\text{sec}}$$

The units given in the table above are of course not the only basic units. For instance, temperature units can be given in either Fahrenheit or Centigrade. In most physics formulas however, absolute or Kelvin temperature scale is used. It starts at absolute zero but the degrees are centigrade degrees. As an example, the gas law (perfect gases) states that:

(1) $PV = u R T$

where P is the pressure, V the volume, u the number of moles of gas, R the universal gas constant and T is the absolute temperature.

(a) An application of the first rule about units is to check that equations are dimensionally correct.

Letting L denote length and T denote time

one could, for instance check out the equation for distance when acceleration is constant.

acceleration is constant.

$d = \frac{1}{2} at^2$ d is distance

a is acceleration
t is time in seconds

Using L, T symbols as above we get

$$L = \left(\frac{L}{T^2}\right) T^2 = L$$

b) Another application is to find units of constants appearing in equations. A first example is the universal gravitation constant appearing in the law of universal gravitation.

$$F = \frac{Gm_1m_2}{d^2}$$

F is the force and its units can be derived most easily from

$F = ma \rightarrow$ units of $F = \frac{ML}{T^2}$

where M is symbol for mass units.

Symbolically:

$$\frac{ML}{T^2} = [G] \frac{M^2}{L^2}$$

[G] is notation for units of G,
the universal gravitation constant

or $[G] = \frac{ML}{T^2} \frac{L^2}{M^2} = \frac{L^3}{MT^2}$

For practical reasons it is easier to give units of G in terms of force units, [F], hence

$$[G] = [F] \frac{L^2}{M^2}$$

Another example would be to derive the units of R from eq. (1).

Conversions

In many practical applications it may be necessary to convert from one set of units to another. This is the type of problem in which it is very easy to make mistakes by multiplying instead of dividing and vice-versa. One of the easiest methods to get the correct result is to use the units as if they were algebraic units and simply rewrite every unit in terms of the new unit. As examples see the following:

(a) 50 ft/sec = ? miles/hour

1 ft

$$\text{(a)}\quad 50\,\frac{\text{ft}}{\text{sec}} = ?\ \frac{\text{miles}}{\text{hour}}$$

$$1\text{ ft} = \frac{1\text{ mile}}{5280} \qquad 1\text{ sec} = \frac{1\text{ hour}}{3600}$$

$$50\,\frac{\text{ft}}{\text{sec}} = 50\,\frac{\frac{1\text{ mile}}{5280}}{\frac{1\text{ hour}}{3600}} = \frac{50 \times 3600}{5280}\ \frac{\text{miles}}{\text{hour}}$$

$$\text{(b)}\quad 50\,\frac{\text{miles}}{\text{hour}} = ?\ \frac{\text{meter}}{\text{sec}}$$

1 mile = 1609 meters 1 hour = 3600 sec

$$50\,\frac{\text{miles}}{\text{hour}} = 50 \times \frac{1609\text{ meters}}{3600\text{ sec}} = \frac{50 \times 1609}{3600}\ \frac{\text{meters}}{\text{sec}}$$

$$= 22.35\,\frac{\text{meters}}{\text{sec}}$$

The purpose here is to avoid any guessing. One should emphasize that it is best to "not skip steps". If the recipe is followed the results are always correct.

LEARNING STRATEGIES – STUDY SKILLS

In many ways, the technical student is not the usual college student. He is not in college because he thirsts for knowledge or wishes to become a scholar in his field. However, he does, in general, know where he wants to go and what he wants to do when he leaves college. He has set his goal for a specific job, a specific employment position. It is towards this goal that all his energies are geared. The technical student is most interested in employability. College courses, including reading and study skills improvement, are secondary – they are a means to a goal. This student works not for knowledge itself but for the job opportunities which knowledge will provide. He is often hesitant to spend time on something if the immediate relevancy isn't apparent. Although the amount of theory known is often deficient, the amount of practical experience is plentiful for the technical student and this often makes him impatient with book or lecture learning; he might even consider it a waste of time. His behavior in the classroom is often influenced by his peers and the chemistry of the class, the interaction between students and between students and teachers becomes very important for the teaching situation. The study techniques of the technical student, especially one with little academic background, are generally primitive – he has no idea HOW to study or even WHERE to begin. He cannot organize his learning or his learning materials.

In the classroom, it is important to recognize and understand the student who sits before you. What kind of academic experiences have gone before? What is the level of his vocabulary? Is he proficient or deficient in vocabulary? What is the level of his reading comprehension? Is it adequate or inadequate? Or, more specifically, is his reading comprehension adequate or inadequate for YOUR course? What is the general readability level of the textbooks which you use? How do these levels compare with the general reading ability levels of the class? If the readability levels are inconsistent, it is possible that these consequences will follow: students don't read the text; if they do read it, they really don't understand it, they cannot absorb and subsequently apply the mathematical concepts.

How does reading influence the study of mathematics? First of all, the student must be able to recognize, comprehend and ultimately apply the technical vocabulary. Most authors of Math textbooks introduce technical vocabulary in context through direct explanation or definition followed by appropriate examples, problems or definitions. While this might be obvious to the mathematics teacher, it is NOT obvious to the student. Point out this technique to him. Gather several examples of this from the textbook and present them to him as a vocabulary exercise. For example:

> "The great generality of analytic methods and formulas is due primarily to the use of directed lines. These are lines on which one direction is regarded as positive and the other as negative."*
>
> What are directed lines?
>
> "The intercepts of a line are the distances from the origin to the points where the line cuts the coordinate axes."
>
> "The intercepts of a line are. . . . ."

After demonstrating to the class, have students open texts and find at least 5 more examples of this type of vocabulary aid. Point out to the student the type of clues which the author may provide, such as, italics, boldface or underlining. Show how this method is consistent throughout

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*Longley, William, Smith, Percey F. and Wilson, Wallace A. *Analytic Geometry and Calculus.* Boston: Ginn & Co., 1960. p. 9.

the book. Often the textbook definition is still bewildering to the student. Help him overcome this problem by "pre-teaching" to overcome vocabulary obstacles. Anticipate the words which will give your students difficulty and present them at the end of the class in preparation for the reading assignment. The presentation may follow this format:

1. "The first property we discuss is the Commutative Property of Addition."†
2. Isolate the word on the chalkboard and divide it into syllables:
   com / mu' / ta / tive
3. Help students to pronounce the word part by part.

Encourage the students to find smaller familiar word parts in long, formidable words. In the word "commutative," a student might recognize the word "commute." Follow up by asking, "What does commute mean? What does a commuter do?" By extending and expanding students' responses, the teacher can develop the correct technical meaning and will have helped the students in two ways: First, by looking for word parts he has helped to change the habits of poor readers who often look at the first few letters of a lengthy word and then dimiss it by saying "too tough for me!" Secondly, by associating the idea of a commuter – one who goes back and forth, one who interchanges a place of abode for a place of business – with the idea of the commutative property, you have given the students a helpful memory aid.

Another indispensable tool to learning difficult vocabulary is the recognition and knowledge of Latin and Greek prefixes, suffixes and roots. One glance at an algebra or geometry text should convince you that the teaching of specific word elements which occur over and over again would be time well spent. Perhaps the first set of elements presented could be the number prefixes mono-, pent-, oct-, dec-, etc. Interchange these prefixes with the base gon and graphically show how figure representations change with the substitution of various prefixes. Let THEM draw the conclusions about number of sides indicated, etc. Do the same type of thing with other frequently-used elements:

-hedron
dihedron
tetrahedron
pentahedron

co or con
coefficient
coordinate
collinear
coplanar

inter
interpolate
intersect
intercept

-mut
permutation
commutative

numer
enumerate
denumerable
numerator

nomin
denominator
binomial
polynomial

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†Osborn, Roger et al. *Extending Mathematics Understanding.* Columbus: Charles E. Merrill Books, Inc. 1963. p. 25.

Prepare a chart such as the one pictured below.

LATIN OR GREEK WORD ELEMENT	GENERAL MEANING
co, con, com, col	with, together
inter	between
mono	one, single
poly	many
peri	around
meter	measure
mut	to change, alter
nomin	name
gon	angle
later	side
equ	equal
angul, angle	sharp
rect	straight, right
lin	line, thread
sec, sect	to cut

Make a list of the terms derived from Latin and Greek which you wish the students to learn. Place each word element with its general meaning on the chart. Then make up a set of cards to correspond with the list of words. Each card will contain the following information for one of the words:

definition
literal meaning
combination of elements
vocabulary word

The cards would be similar to the following sample:

"the distance around the outer
boundary of a surface or figure"

perimeter
peri + meter
around to measure

Give one of these cards to each of the students and have them challenge each other to make a word which fits the definition which they read by combining two or more of the word elements on the chart. Completion exercises such as the following are also helpful:

> "Asymptotes are lines which extend infinitely. Although they approach nearer to a curve than any other line, they never meet. "Never meet" comes from the two prefixes $\underset{(a)}{\ldots}$ and $\underset{(sym)}{\ldots}$"*

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*Davis, Nancy B. *Basic Vocabulary Skills*. New York: McGraw Hill Book Company. 1969.

"To indicate the action of changing the sign in an equation and putting it across to the other side, one uses the term . . . . . . pose."†
(trans)

Encourage students to learn technical vocabulary as it appears in their textbooks. If the author provides boldface or italics as a study aid, be sure the students are aware of the significance of these terms. Suggest that they follow the procedure listed below for learning difficult terms:

Tips for Learning Difficult Technical Terms**

1. ATTEND TO EACH TERM WHEN FIRST IT APPEARS. Read reflectively to grasp what the definition is saying – not to memorize by rote but to gain a real appreciation of the meaning.
2. TAKE THE NEW WORD APART IF YOU CAN. Do you recognize a familiar part? If, for example, you recognize the familiar prefix "poly-," meaning "many," you already have a hold on "polynomial," "polygon," and "polyhedron." If you recognize the word part "equi-," meaning "equal," it helps you unlock "equidistant, equiangular, equivalent," and "equation." The familiar prefix "co-," meaning "with" or "together with," can help you master "coordinate, collinear, cosine," and "cotangent."
3. READ AND REREAD AS OFTEN AS NECESSARY. Reading-once-straight-through patterns are not appropriate. Complete stops are called for frequently. Thought time is essential in addition to reading time.
4. The authors' definition of a new term is almost always followed by examples. EXAMINE THESE EXAMPLES CRITICALLY and figure out whether in fact they do follow the definition. If examples are not given, try to create some of your own.
5. TRY TO THINK OF COUNTER EXAMPLES, examples which do not come under the definition. In arriving at these, you may find it helpful to change a word or two in the definition.
6. READ THE DEFINITION, as you read all mathematics, *pencil in hand.* Make jottings and create your own examples.
7. Suppose, as you're reading the definition of the new term, you encounter a technical term you've already met in the course whose meaning now escapes you. We all forget! You have the meaning right at your fingertips through the index of your book. USE THE INDEX for instant access to the original explanation of the forgotten term.
8. As you're working with the new term, try to EXPRESS ITS MEANING in actual words – your own words.
9. You may find a "List of Some Important Terms to Learn" toward the end of each chapter. You'll want to CHECK YOUR UNDERSTANDING of this list of terms. The terms the authors have selected for this list are crucial. You may also want to MAKE YOUR OWN LIST of key terms and their meaning.
10. Make an effort to USE YOUR NEW MATHEMATICAL TERMS.

IN SUMMING UP

1. Read reflectively.
2. Look for familiar word parts.
3. Reread.
4. Scrutinize the examples.
5. Make up counterexamples.
6. Be active with a pencil.
7. Use your index.
8. Self-recite.
9. Review.
10. APPLY YOUR NEW LEARNINGS.

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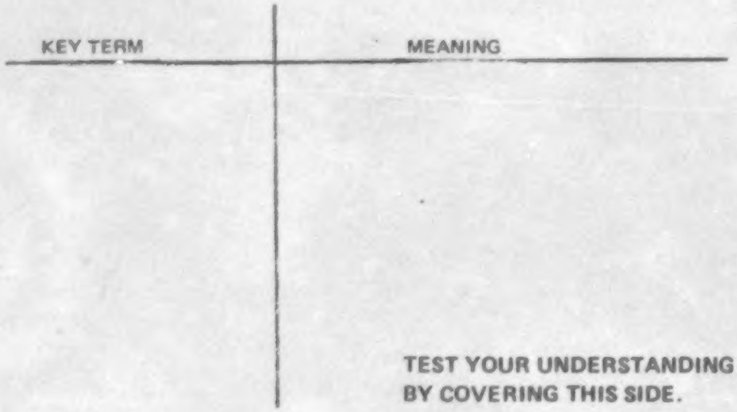
†Ibid.

**Thomas, Ellen Lamar and Robinson, Alan H. *Improving Reading in Every Class.* Boston: Allyn and Bacon. 1972. p. 299.

Another helpful study technique for remembering mathematics vocabulary is to have the student set aside a special section of the notebook as a Mathematics Glossary." As each new term is learned, it is added to the list with its definition of meaning. The student divides the page in half in the following manner:



The student enters the new word on the left under key term and the precise meaning on the right. When review time comes, the divided page will be handy in learning to recite the meaning of terms without looking at the definitions. The student should indicate the end of each unit or chapter, perhaps with a double line, so that if he or she wishes to check on the meaning of words in a particular unit, he or she will know just where to find them.* A similar study technique is to place each new term on a separate index card with the term on one side and the definition on the other. Students quiz themselves by looking at the term and reciting aloud the definition. Later, they reverse the procedure and quiz themselves by reading the definition and reciting the term aloud. Be sure they frequently "shuffle the deck" so that a word is not given a false association through a consistent sequence. This technique can be done by two students or by several students utilizing a round robin procedure.

A major complaint heard from teachers of mathematics is that their students don't read the textbook. How can the math teacher help students to read the text and get something out of it? How can students of math learn to become more independent of the teacher and rely on their own abilities to gather and learn information presented in the textbook? To many students, a math textbook is quite formidable; it assumes that the students have mastered all of the following reading skills in the field of mathematics:

1. Ability to read and understand the technical vocabulary.
2. Ability to recognize and understand algebraic symbols, and letters standing for unknowns; negative and positive signs and operational symbols used in number work.
3. Ability to understand geometrical concepts and generalizations, including their explanation and application, understanding of axioms, postulates, theorems and corollaries.
4. Ability to comprehend and to work with expressions of mathematical relationships – formulas equations and graphs.

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*Thomas, Ellen Lamar and Robinson, Alan H., *Improving Reading in Every Class.*

5. Ability to read figures, diagrams and graphs and to see the relationship between different parts.
6. Ability to read and solve reasoning problems.
7. Ability to find proofs of statements related to the solution of problems.
8. Following a plan for solving math problems.

Just how well will your students handle the textbook which you have selected for the course? At the beginning of the term, try to assess the reading competencies of your students by assigning a passage in the text which is typical of the independent reading that you expect they will do during the course. Direct them to read and study the passage in class just as if they were studying for or preparing for a test on what it says. Explain that in a short while they will be quizzed on the contents and indicate that they may use any scratch sheets to take notes or make jottings if they wish. Students who do not finish reading the passage in the allotted time should indicate the point to which they read. All notes and scratch sheets are handed in as well as answers to the quiz. By studying these results and observing the students while at work, several insights can be gained:

1. Can the student handle the textbook, or does it appear to be beyond him?
2. Can he or she master clearly explained technical terms independently? grasp key concepts? get the message of diagrams and figures?
3. Does he or she use a scratch sheet to study actively – to jot down important ideas, make his or her own sketches, fill in the inner steps of explanations?
4. Does he or she appear to be an extremely slow reader?*

Once you have some idea of the abilities of your students, spend time on a "meet your textbook session," pointing out the most important features and study aids which can be found in the book. The following items (taken from Lamar and Robinson, page 309) shoud be covered:

The table of contents with its concise, sequential listing of major topics covered
Lists of mathematical symbols for easy reference
Large-size or boldface headings that announce the content of a section
Italics, boldface or color used to signal "official" terms
Italics, boldface, or color used to call attention to concepts, rules, or principles that should be learned and to flag these for easy reference
Typographical danger signals of pitfalls to avoid
Aids for pronouncing and accenting difficult new terms (if these aids are present)
Chapter summaries that wrap up big ideas
Self-check tests at the close of chapters
Table of squares and square roots
Reference list of axioms
The glossary
The index

In addition, have a "how to read your textbook" session. Demonstrate the following helpful procedures which are outlined by Ellen Lamar Thomas and H. Alan Robinson in their book, **Improving Reading In Every Class.**

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*Lamar and Robinson, p. 307.

I. Pre-read to look over the material

A. Read through the passage once at a moderate speed. If necessary, slow down to get the gist of a paragraph but remember that this is not the in-depth reading.

B. Read with a questioning mind set. Ask yourself:

1. How does this relate to what I've been studying?

2. What's the author driving at? Where is this all leading?

3. What should I look for when I go back and really study this section?

II. In-depth Reading

A. Set a purpose for reading by turning subheadings or titles into questions.

B. Read to answer those questions.

C. Read to pull out the meaning of each word and phrase and sentence. Show how even the smallest word cannot be slighted; noticing that "or" instead of "and" is crucial and observing the phrase "at least one," not "exactly one," is often critical.

D. Remember to use textbook aids such as the index for forgotten meanings and terms.

E. Complete stops are often called for. As you read reflect on what the author says. When the author poses a question, try to answer it. If he states "it is obvious," see if it is obvious to you.

F. Study carefully all diagrams and figures and understand the relationship between them and the concepts which are presented in the reading.

1. Shift your eyes from text to diagram as needed.

2. Test your understanding by covering the text and provide an explanation by looking at the diagra alone.

3. If you cannot provide an explanation as stated in No. 2 above, then reread, referring back and forth to the figure until you are sure you understand it.

4. Try to conceal both explanation and diagram and attempt to visualize the figure or sketch it on a piece of paper.

III. Take Notes or Use a Scratch Sheet as you Read

A. Capsulize important ideas. Force yourself to repeat difficult passages in your own words.

B. Fill in inner steps.

C. Answer the author's questions.

D. Sketch constantly as you study.

IV. Final Reading – Reread the material quickly so that you will not lose the overall thought amongst the details and specific explanations.

After a class demonstration with active student participation, the reading of the textbook should no longer be a problem and chore for the students. However, this method must be consistently applied if it is to yield real results. If individual students still complain that they "just can't get it," suggest these final study aids:

1. Reread – successful math students read explanations again and again. A passage that blocked you at first encounter may come clear with a fourth or fifth reading.
2. Restate – Trying to re-express an idea forces you to concentrate on what is being said – it may clinch your understanding.
3. Read on – If after repeated efforts the meaning still escapes you, read on in the assignment and then return later. The difficulty may clear up in the light of the rest of the passage.
4. Try again after time has passed – A difficult passage often hits us differently later.
5. Search out other books – The corresponding explanation in other mathematics textbooks is often helpful. You would naturally expect several explanations to be clearer than a single explanation.
6. Consult a friend – Let a friend help you, not tell you.
7. Not a single question should remain unanswered – jot down questions to bring to class or conference.*

The field of education is rich with theories, techniques, and methods to the point of confusion. Let us consider some basic principles, as related to the student population under consideration.

Often the technical student is separated from the one in engineering along a number of dimensions. One such distinction is related to level of abstraction, another is responsibility. Yet, many of the two-year technical graduates continue to a four-year degree or more, and, depending on the individual person and position, technical graduates may move to positions of responsibility. Even within the accepted scheme of responsibility hierarchy, the technician may act as a translator-mediator bridging between the general-abstract level and the specific-concrete one of the machinist. As such, the technician needs to understand and be versed in both levels. The technician must be well-rooted in the fundamentals so that if willing and able, can develop from a problem-solver to a point of being able to evaluate, be creative and resourceful beyond the basic factual information gathered during his schooling.

"Good teachers" are as varied as their personalities; they are not all the same. Yet, they do have a number of points in common: intellectual capacity, curiosity and knowledge of the material, and sincere interest in the students.

Fred C. Morris† compiled a self critique list covering the planning, the class session and testing phases of the teaching activity – and it is reproduced below.

Lesson Plan

1. Do you plan your lesson, or do you go to class with only a general idea of what you are going to do?
2. Is the objective well defined, and can it be accomplished by the presentation that you plan to make?
3. Do you study each particular topic to find the most effective way to present it?
4. Does your lesson plan include all of the important points and exclude the irrelevant?
5. Is your lesson plan logical in order and does it make a clear connection with what has gone before?
6. Does your lesson plan have application to some specific thing that the students are to do?
7. Do you make an intelligent and effective use of instructional aids?

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*Lamar and Robinson, p. 293.

†Morris, Fred C. *Effective Teaching: A Manual for Engineering Instructors.* New York: McGraw-Hill Book Co., 1950.

Conduct of Class Session

1. Do you know what you are trying to do in class?
2. Do you tell the students what you are talking about and why?
3. Do you spend most of the time just talking? If you do, the chances are that the students do not learn much.
4. Do you have the class session organized so that there is no waste of time?
5. Do you control the thinking of the students so as to gain and hold their interest?
6. Are you prepared for the questions asked, or do you evade them?
7. Are you courteous and considerate in answering student questions?
8. Do you use some inappropriate instructional aid because you do not know what else to do, or because it is easier than preparing an effective presentation?
9. Do you use an unreasonable amount of the students' time in copying notes from the board which you could give out in mimeographed form?
10. Are you well-groomed, neat, and clean? The bandbox effect is not desirable, but you should be presentable to polite society.
11. Do you use annoying mannerisms of person or speech which distract the students?
12. Do you display a sense of humor? You should, of course, move along with your job of teaching, but this can be done without acquiring the atmosphere of a morgue.
13. Do you hide the blackboard with your body, and do you face the class or talk to the blackboard?
14. Do you start and end your class session on time?
15. Do you really teach the students something, or merely confuse them on the subject?
16. Do the students go to sleep in class? If they do, it is time for you to wake up.

Testing

1. Do you go to the trouble to construct good tests?
2. Are your tests representative of the material given in class, clear as to meaning, and of reasonable length?
3. Do you let your class go without giving tests because you do not like to grade the papers?
4. In grading tests, do you give each question a value in proportion to its importance?
5. Are you always fair and impartial in grading?
6. Do you have sufficient information on each student to establish fair term grades?
7. Do you fail some definite percentage of every class? This practice is unsound, unfair, and has a devastating effect upon student morale.
8. Do you hide poor teaching with high grades?
9. Do you give back the test papers at the following class session, or do you wait until the students ask about them several times?
10. Do you realize that every time you give the students a test, you are testing yourself too? If you have a large number of failures, you had better examine your teaching methods.

Students, as a group, exhibit various traits and have different expectations, changing somewhat from campus to campus, and from year to year. However, most students in the technology field tend to be goal-directed and less floundering, and more likely to have been exposed to their area of interest as compared with most other groups. Such a statement may, of course, vary between specific individuals. Joseph Zimmerman* conducted a survey at the Worcester Polytechnique Institute identifying some of the items that "turn students on and off."

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*Zimmerman, Joseph F., "What Motivates Students? A Second Look." *Journal of Engineering Education*, October 1964, vol. 55, no. 2, pp. 53-55.

TABLE I. ITEMS FROM QUESTIONNAIRE

Item	*PERCENT (total Group)* 1955	1964
Positive		
Practical value of the course in earning a living	36	37
Instructor's knowledge of subject and related fields	30	37
Well-defined course objectives	25	14
Enthusiasm of the instructor	25	38
Instructor's sympathy & Understanding of students & their problems	24	13
Instructor's willingness to answer questions	24	9
Knowledge of your progress	23	20
Grades	23	46
Emphasis on fundamentals rather than details	22	14
Courteous treatment of instructor	14	3
Negative		
Instructor's use of sarcasm	66	64
Instructor's use of fear	55	53
Instructor's use of self-approval	45	10
Personal cross-examination by the instructor	26	30

Now, see how many of these points apply to you, the individual teacher, to your classroom, and to your campus, and where you can maximize the teaching process. Last but not least, do not hesitate to consult your colleagues and if necessary the counseling services on your campus.

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